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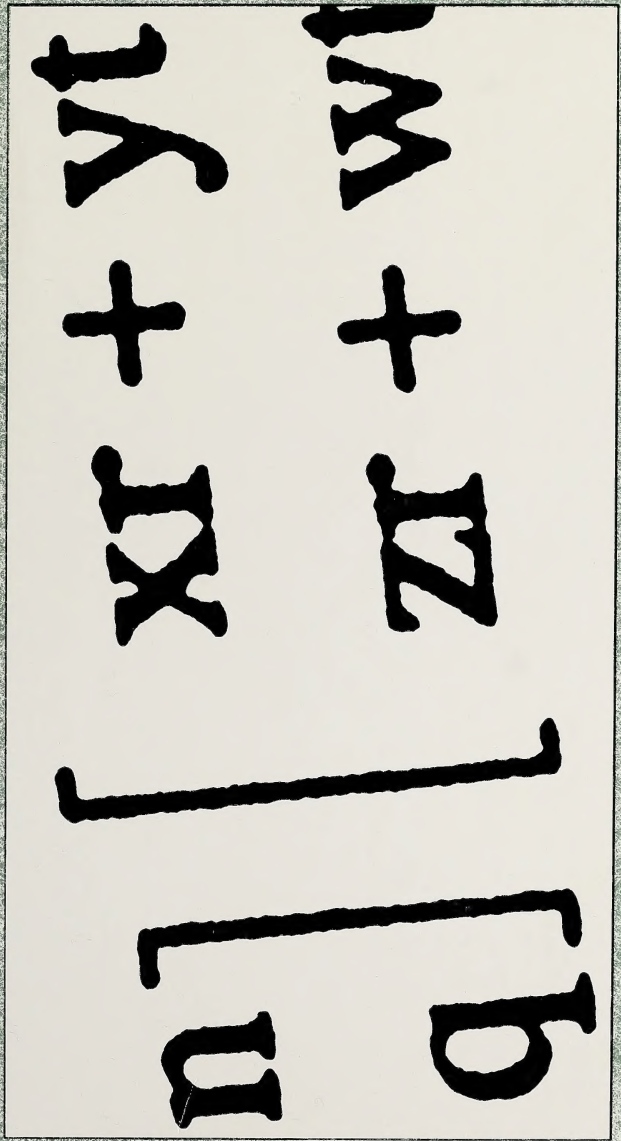
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MATHEMATICS 3!

Distance
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UNIT 9: MATRICES AND LINEAR TRANSFORMATIONS

Alberta
EDUCATION





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W e l c o m e



Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 31 Student Module Unit 9 Matrices and Linear Transformations Alberta Distance Learning Centre ISBN No. 0-7741-0444-9

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General Information

This information explains the basic layout of each booklet.

- **What You Already Know and Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.
- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.
- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.
- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.
- **Extensions** gives you the opportunity to take the topic one step further.
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.
- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics (**Appendix B, etc.**).

Visual Cues

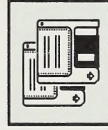
Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

An explanation of what they mean is written beside each visual cue.



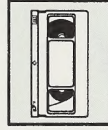
Audiotape

- learning by listening to an audiotape



Computer Software

- learning by using computer software



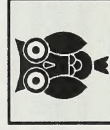
Videotape

- learning by viewing a videotape



Print Pathway

- choosing a print alternative



What You Already Know

- reviewing what you already know



Review

- studying previous concepts



Introduction

- introducing the unit



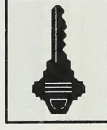
What Lies Ahead

- previewing the unit



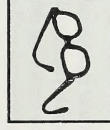
Exploring the Topic

- actively learning new concepts



Key Idea

- flagging important ideas



Another View

- exploring different perspectives



Solutions

- correcting the activities



Extra Help

- providing additional study



Extensions

- going on with the topic



What You Have Learned

- summarizing what you have learned

Mathematics 31

Course Overview

Mathematics 31 contains 9 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1	10%
Introduction to Differential Calculus	
Unit 2	10%
Differentiation of Algebraic Expressions and Graphing	
Unit 3	20%
Practical Application of Derivatives	
Unit 4	10%
Integration	
Unit 5	10%
Geometric Vectors and Their Application	
Unit 6	10%
Algebraic Vectors and Their Application	
Unit 7	10%
Inner Product	
Unit 8	10%
Systems of Linear Equations	
Unit 9	10%
Matrices and Linear Transformations	
	100%

Unit Assessment

After completing the unit you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Matrices and Linear Transformations

This unit covers topics dealing with matrices and linear transformations. Each topic contains explanations, examples, and activities to assist you in understanding matrices and linear transformations. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in **Appendix A**. In several cases there is more than one way to do the question.

Unit 9 Matrices and Linear Transformations

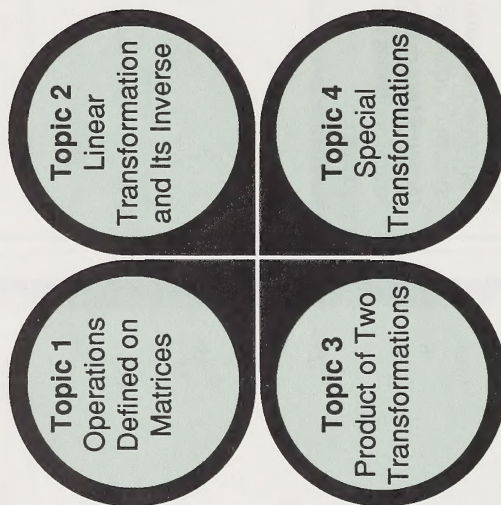
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Matrices and Linear Transformations

In the previous unit you studied systems of linear equations. A system of equations produces a transformation, and it can be denoted by a matrix. Systems of equations, matrices, and linear transformations are closely related. You are going to study these in this unit.

Unit 9 Matrices and Linear Transformations





What You Already Know

Refresh your memory!

Recall the following:

- the three methods used to solve a system of linear equations with three unknowns are
 - the comparison method
 - the substitution method
 - the elimination method
- augmented matrices of a system of linear equations and its row-reduced echelon form
- solving systems of linear equations using row-reduced echelon form

Now that you have looked at topics that you have previously studied, go to the **Review** to confirm your understanding of this material. If you had trouble with this material, you may need to review **Unit 8**.



Review

Try the following review questions.

1. Solve the following system of equations.

$$\begin{array}{rcl} 3x + 5y - 6z = 1 & \textcircled{1} \\ x + y - z = 2 & \textcircled{2} \\ x - 2y + z = 3 & \textcircled{3} \end{array}$$

2. Write out the augmented matrix for the following system of equations, and find its row-reduced echelon form.

$$\left\{ \begin{array}{l} x - y + 2z = 4 \\ 2x + y - z = 7 \\ x + 2y + z = 5 \end{array} \right.$$

3. Use the row-reduced echelon form obtained in question 2 to solve the system of equations.



Now go to the **Review** solutions in **Appendix A**.

Topic 1 Operations Defined on Matrices



Introduction

Can you add, subtract, and multiply matrices? In this topic you are going to learn the arithmetic of matrices – specifically, the fundamental processes of matrix addition and matrix multiplication. You will learn the properties of addition of matrices, how and under what circumstances matrix multiplication is possible, and the algebraic properties of multiplication of matrices.



What Lies Ahead

Throughout this topic you will learn to

1. add and subtract matrices
2. multiply matrices

Now that you know what to expect, turn the page to begin your study of operations defined on matrices.



Exploring Topic 1

Activity 1



Add and subtract matrices.

In Unit 8 the definition was given for an $m \times n$ real matrix. A 2×3 (read 2 by 3) real matrix is a rectangular array of real numbers such as this:

$$\begin{bmatrix} 2 & 0 & -5 \\ 1 & 3 & 7 \end{bmatrix}$$

The number of rows (m) is stated followed by the number of columns (n). An $m \times n$ matrix is said to have dimensions $m \times n$. In other words the dimensions of a matrix are given by stating the number of rows followed by the number of columns in the matrix. If two matrices have the same dimensions, then the two matrices would have the same number of rows and the same number of columns.

Some examples of matrices and their respective dimensions are as follows:

$$\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

The dimensions are 2×2 .

$$\begin{bmatrix} 3 & 0 & 1 & 4 \end{bmatrix}$$

The dimensions are 1×4 .

$$\begin{bmatrix} 5 & 4 \\ 3 & 1 \\ 2 & 2 \\ 0 & 7 \end{bmatrix}$$

The dimensions are 4×2 .

Sometimes you would like to talk about a matrix and its elements without any specific number in mind. You would have to establish a convention to determine the position of elements in a matrix. You can use a single letter with a set of double subscripts (such as a_{rc}) to represent the number at a specific position.

Use a_{rc} where r represents the row number and c represents the column number.

Consider the following matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

The real number in row 1, column 1 is represented by a_{11} .
 The real number in row 1, column 2 is represented by a_{12} .
 The real number in row 1, column 3 is represented by a_{13} .
 The real number in row 2, column 1 is represented by a_{21} .
 The real number in row 2, column 2 is represented by a_{22} .
 The real number in row 2, column 3 is represented by a_{23} .

The row is represented by the subscript r , and the column is represented by the subscript c .

Note that a_{rc} is not necessarily a real number. However, in this course you will use real matrices in which all the components are real numbers.

Capital letters (A , B , C , etc.) are often used to indicate certain matrices.

The following examples will help you to understand the definitions.

Example 1

Identify a_{12} , a_{21} , and a_{22} in the matrix $\begin{bmatrix} 3 & 5 & 7 \\ 8 & 0 & 4 \end{bmatrix}$.

Solution:

$$a_{12} = 5$$

$$a_{21} = 8$$

$$a_{22} = 0$$

Example 2

If $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 7 \\ -1 & -5 \end{bmatrix}$, find the following.

$$\bullet a_{21}b_{11} + a_{11}b_{21}$$

Solution:

$$a_{21} = 4$$

$$b_{11} = 0$$

$$a_{11} = 3$$

$$b_{21} = -1$$

$$\begin{aligned} \therefore a_{21}b_{11} + a_{11}b_{21} &= 4(0) + 3(-1) \\ &= -3 \end{aligned}$$

$$\bullet a_{11}b_{22} + a_{22}b_{11}$$

Solution:

$$a_{11} = 3$$

$$b_{22} = -5$$

$$a_{22} = 6$$

$$b_{11} = 0$$

$$\begin{aligned}\therefore a_{11}b_{22} + a_{22}b_{11} &= 3(-5) + 6(0) \\ &= -15\end{aligned}$$

If A and B are $m \times n$ matrices and $a_{rc} = b_{rc}$, then $A = B$.



If two matrices have the same dimensions and the corresponding elements are equal, the two matrices will be considered equal.

Example 3

Can $A = B$, $B = C$, or $C = A$ if $A = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$, and

$$C = \begin{bmatrix} 5 & x \\ 1 & y \end{bmatrix}?$$

Solution:

$A \neq B$ because A and B do not have the same dimensions;
 $B \neq C$ because B and C do not have the same dimensions; and
 $A = C$ if and only if $x = 6$ and $y = 8$.



It is possible to add two matrices if they have the same dimensions. They can be added to one another by adding their corresponding elements. If they do not have the same dimensions, then they are not compatible with respect to addition and the process of addition will be meaningless.

The following example will demonstrate how to add two matrices.

Example 4

Find $A + B$ if $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & -1 \end{bmatrix}$.

Solution:

$$\begin{aligned}A + B &= \begin{bmatrix} 2+0 & 3+1 & 5+2 \\ 1+3 & -2+4 & 1+(-1) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 7 \\ 4 & 2 & 0 \end{bmatrix}\end{aligned}$$

In Example 4 the solution is a 2×3 matrix. It shows that the set of real matrices is **closed under addition**.

Now that you have defined the addition of matrices, you may want to denote expressions such as the sum of $A + A$ by $2A$ and $A + A + A$ by $3A$. What does this mean in terms of the matrix elements? $A + A$ means you are adding all the corresponding elements of two identical matrices. Since all the corresponding elements are equal, the result of elements would be two times the original. It would be convenient to define multiplication of a matrix by a scalar.

If c is a real number and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

$$\text{then } cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}.$$

This definition may be extended to the product of a real number and an $m \times n$ matrix.

Study the following example.

Example 5

Find $3A$ and $-5A$ if $A = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$.

Solution:

$$3A = \begin{bmatrix} 3(0) & 3(4) \\ 3(1) & 3(3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 12 \\ 3 & 9 \end{bmatrix}$$

$$-5A = \begin{bmatrix} -5(0) & -5(4) \\ -5(1) & -5(3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -20 \\ -5 & -15 \end{bmatrix}$$

The concept used in matrix addition is different from ordinary addition. Would familiar properties (such as the associative property) hold for all matrices under the operation of addition? Although the proofs of these properties are very straightforward, you are not going to prove any of these here.

The multiplication of a matrix by a scalar is represented by $2A$ and $3A$.

A scalar is a real number rather than a vector.

The closure property was illustrated in Example 4. The next property to discuss is the **commutative property**.

Example 6

Find $A + B$ and $B + A$ if $A = \begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -7 \\ 2 & 4 \end{bmatrix}$.

Solution:

$$\begin{aligned} A + B &= \begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & -7 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 5+0 & 3-7 \\ 1+2 & -2+4 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -4 \\ 3 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} B + A &= \begin{bmatrix} 0 & -7 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0+5 & -7+3 \\ 2+1 & 4-2 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -4 \\ 3 & 2 \end{bmatrix} \end{aligned}$$

Notice that $A + B = B + A$. This result is true in general. Therefore, addition of $m \times n$ real matrices is commutative.

The following example will show the **associative property**.

Example 7

Find $(A + B) + C$ and $A + (B + C)$ if $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 5 & 3 \end{bmatrix}$, and

$$C = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}.$$

Solution:

$$\begin{aligned} (A + B) + C &= \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 5 \\ 7 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 7 \\ 8 & 8 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A + (B + C) &= \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 6 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 7 \\ 8 & 8 \end{bmatrix} \end{aligned}$$

Notice that $(A+B)+C = A+(B+C)$ for the matrices of this example. Actually, addition of any $m \times n$ real matrices is associative.

In the set of real numbers, 0 is the **additive identity**. Is there a zero matrix in the set of real matrices? What is a zero matrix? All the elements of a zero matrix are zeros. A 2×2 zero matrix would look like the following:

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following example shows that a zero matrix works like the real number 0.

Example 8

Find $A+O$ if $A = \begin{bmatrix} 2 & 3 \\ -2 & 5 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution:

$$\begin{aligned} A+O &= \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2+0 & 3+0 \\ -1+0 & 5+0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \end{aligned}$$

Therefore, the zero matrix is the additive identity in the set of matrices.

So far only addition has been discussed. However, the same rules for addition apply to matrix subtraction. If A and B are two matrices, then $A-B = A+(-B)$. The two matrices must have the same dimensions in order to subtract the corresponding elements. Look at the following example.

Example 9

Find $A-B$ if $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$.

Solution:

$$\begin{aligned} A-B &= A+(-B) = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -1 & -2 \\ 0 & -1 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 2-3 & 3-1 & -1-2 \\ 1-0 & 0-1 & 4-5 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 2 & -3 \\ 1 & -1 & -1 \end{bmatrix} \end{aligned}$$

If A is a matrix, then $A+(-A) = O$ because the corresponding elements are equal; thus, the difference must be a zero matrix. Therefore, call $(-A)$ the **additive inverse** of A .

Try the following questions. (Do any six.)

1. List a_{21} , a_{12} , and a_{32} for the following matrices.

a. $\begin{bmatrix} 3 & 7 \\ 5 & 9 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 3 & 8 \\ 1 & -2 & 4 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 0 & -3 \end{bmatrix}$

2. State the dimensions of the matrices in question 1.

3. Find a and b if $A = B$ where $A = \begin{bmatrix} 3 & 4 & 0 \\ a & -2 & -1 \end{bmatrix}$ and

$$B = \begin{bmatrix} 3 & 4 & 0 \\ 5 & -2 & b \end{bmatrix}.$$

4. If $A = \begin{bmatrix} 2 & 3 & -5 \\ 1 & -6 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 4 \\ 1 & -3 & -2 \end{bmatrix}$, then find the following.

a. $a_{23} + a_{11}b_{22}$

b. $a_{13}b_{23} + a_{22}b_{21}$

5. Find the sum of matrices A and B in question 4.

6. If $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 0 \\ 5 & -3 \end{bmatrix}$, find the following.

a. $A + B$

b. $B - C$

c. $-3B$

d. $\frac{1}{2}A$

e. $A - [3B + C]$

7. Verify the following if $A = \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$, and

$$C = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}.$$

- a. the commutative property of addition
($A + B = B + A$)
- b. the associative property of addition
[$(A + B) + C = A + (B + C)$]

8. Find $2A - B - 3C$ if $A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & -2 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & -2 \end{bmatrix}$.



For solutions to Activity 1, turn to Appendix A, Topic 1.

Activity 2



Multiply matrices.

Multiplication of matrices is not like multiplication of real numbers.



Matrix multiplication is defined only when the number of columns of the multiplier equals the number of rows of the multiplicand.

In other words, an $m \times n$ matrix can be multiplied by an $n \times p$ matrix if the $n \times p$ matrix has n rows. If the number of columns of the first matrix is not equal to the number of rows of the second matrix, then no product matrix can be obtained.

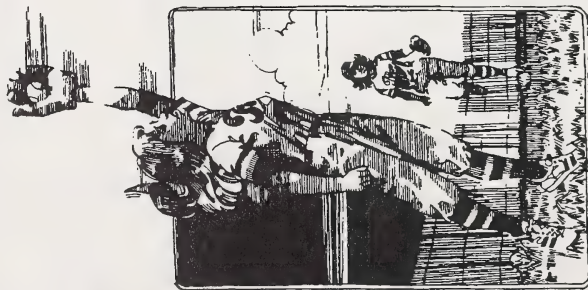
The following demonstration will illustrate how multiplication of the following two 2×2 general matrices can be done.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, the

product $C = AB$.

The first matrix has n columns and the second matrix has n rows.

For the product AB , A is the multiplier and B is the multiplicand.



$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$\text{Let } C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

$c_{11} = a_{11}b_{11} + a_{12}b_{21}$ is obtained by multiplying the corresponding elements of the first row of the first matrix and the first column of the second matrix and then adding the results.

$c_{21} = a_{21}b_{11} + a_{22}b_{21}$ is obtained by multiplying the corresponding elements of the second row of the first matrix and the first column of the second matrix and then adding the results.

$c_{12} = a_{11}b_{12} + a_{12}b_{22}$ is obtained by multiplying the corresponding elements of the first row of the first matrix and the second column of the second matrix and then adding the results.

$c_{22} = a_{21}b_{12} + a_{22}b_{22}$ is obtained by multiplying the corresponding elements of the second row of the first matrix and the second column of the second matrix and then adding the results.

If you find these terms confusing, the following example may provide you with a better understanding.

Example 10

$$\text{Find } AB \text{ if } A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & -2 \\ -1 & 4 \end{bmatrix}.$$

Solution:

$$\begin{aligned} AB &= \begin{bmatrix} 2(7) + 1(-1) & 2(-2) + 1(4) \\ 3(7) + 5(-1) & 3(-2) + 5(4) \end{bmatrix} \\ &= \begin{bmatrix} 13 & 0 \\ 16 & 14 \end{bmatrix} \end{aligned}$$

In general if A is an $m \times n$ matrix with rows A_1, A_2, \dots, A_m and B is an $n \times p$ matrix with columns B_1, B_2, \dots, B_p , then

$$C = AB = \begin{bmatrix} A_1 B_1 & A_1 B_2 & \dots & A_1 B_p \\ A_2 B_1 & A_2 B_2 & \dots & A_2 B_p \\ \vdots & \vdots & \ddots & \vdots \\ A_m B_1 & A_m B_2 & \dots & A_m B_p \end{bmatrix}.$$

C will be an $m \times p$ matrix.

Study the following example.

Multiplication of matrices is often called row-by-column multiplication.

$$\begin{aligned} \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix} &= 13 & \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} &= 0 \\ \begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix} &= 16 & \begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} &= 14 \end{aligned}$$

Use a_{11}, a_{12}, \dots to represent elements of matrix A .

Use A_1, A_2, \dots, A_m to represent the rows of matrix A and B_1, B_2, \dots, B_p to represent the columns of matrix B .

Example 11

Find AB if $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 & 5 & 0 \\ -2 & -3 & -4 & 1 \end{bmatrix}$.

Solution:

$$\begin{aligned} C = AB &= \begin{bmatrix} 1(-1)+0(-2) & 1(3)+0(-3) & 1(5)+0(-4) & 1(0)+0(1) \\ 2(-1)+4(-2) & 2(3)+4(-3) & 2(5)+4(-4) & 2(0)+4(1) \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 & 5 & 0 \\ -10 & -6 & -6 & 4 \end{bmatrix} \end{aligned}$$

Note that A is a 2×2 matrix, B is a 2×4 matrix, and C is a 2×4 matrix.

If A is a 2×3 matrix and B is a 3×4 matrix, then the product AB produces a matrix C that has dimensions 2×4 .

$\underbrace{\hspace{1.5cm}}_{\text{matrix } C \text{ is } 2 \times 4.}$
matrix A is 2×3 . matrix B is 3×4 .

When you do this type of question, you have to be careful that you do not mix up the rows and the columns.

Now do questions 1 to 3 in the following exercise to practise this method. If no product matrix can be obtained, write **impossible**.

1. If $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, find AB and BA .

2. Find AB and BA if $A = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$ and

$$B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ -1 & 5 \end{bmatrix}.$$

3. If $A = \begin{bmatrix} 2 & 1 & -3 \\ 0 & -1 & -2 \end{bmatrix}$ and

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & -1 & 5 \\ -2 & 1 & -4 & 7 \end{bmatrix}, \text{ find } AB.$$



For solutions to Activity 2, turn to **Appendix A, Topic 1**.

Now that you can multiply matrices, it is time to examine the algebraic properties of multiplication of matrices. Look at the **closure property** first. It is obvious that the product of any two 2×2 matrices is a 2×2 matrix. You can say that the set of 2×2 matrices is **closed under multiplication**. As mentioned it is not

always possible to multiply any two matrices. The closure axiom holds for square matrices only. The product of two $m \times m$ matrices is an $m \times m$ matrix.

The other properties of square matrices will now be examined.

The next property to examine is the **commutative property**. If A and B are square matrices, is $AB = BA$? For some special matrices this may be true, but in general this is not always true.

For example, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & -1 \\ -2 & -3 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1(0) + 2(-2) & 1(-1) + 2(-3) \\ 3(0) + 4(-2) & 3(-1) + 4(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -7 \\ -8 & -15 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0(1) + (-1)(3) & 0(2) + (-1)(4) \\ (-2)(1) + (-3)(3) & (-2)(2) + (-3)(4) \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -4 \\ -11 & -16 \end{bmatrix}$$

Therefore, $AB \neq BA$ and multiplication of matrices is **not** commutative.

Now look at the associative property. If A , B , and C are square matrices, is $A(BC) = (AB)C$?

For example, $A = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$, $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, and $C = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$.

$$\begin{aligned} A(BC) &= \begin{bmatrix} m & n \\ p & q \end{bmatrix} \left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix} \right) \\ &= \begin{bmatrix} m & n \\ p & q \end{bmatrix} \begin{bmatrix} xr + yt & xs + yu \\ zr + wt & zs + wu \end{bmatrix} \\ &= \begin{bmatrix} mxr + myt + nzt + nwt & mxs + myu + nzs + nwu \\ pxt + pyt + qzt + qwt & pxs + pyu + qzs + quu \end{bmatrix} \end{aligned}$$

$$\begin{aligned} (AB)C &= \left(\begin{bmatrix} m & n \\ p & q \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \right) \begin{bmatrix} r & s \\ t & u \end{bmatrix} \\ &= \begin{bmatrix} mx + nz & my + nw \\ px + qz & py + qw \end{bmatrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix} \\ &= \begin{bmatrix} mxr + myt + nzt + nwt & mxs + myu + nzs + nwu \\ pxt + pyt + qzt + qwt & pxs + pyu + qzs + quu \end{bmatrix} \end{aligned}$$

Therefore, $A(BC) = (AB)C$, and the associative property holds for multiplication of square matrices.

Consider the following example.

Example 12

Show that $A(BC) = (AB)C$ if $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -1 \\ 0 & 4 \end{bmatrix}$, and

$$C = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}.$$

Solution:

$$\begin{aligned} A(BC) &= \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \left(\begin{bmatrix} -2 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2(1) + (-1)(3) & (-2)(1) + (-1)(5) \\ 0(1) + 4(3) & 0(1) + 4(5) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -5 & -7 \\ 12 & 20 \end{bmatrix} \\ &= \begin{bmatrix} -5 + 0 & -7 + 0 \\ -15 + 24 & -21 + 40 \end{bmatrix} \\ &= \begin{bmatrix} -5 & -7 \\ 9 & 19 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} (AB)C &= \left(\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 0 & 4 \end{bmatrix} \right) \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -2 + 0 & -1 + 0 \\ -6 + 0 & -3 + 8 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -2 - 3 & -2 - 5 \\ -6 + 15 & -6 + 25 \end{bmatrix} \\ &= \begin{bmatrix} -5 & -7 \\ 9 & 19 \end{bmatrix} \end{aligned}$$

Therefore, $A(BC) = (AB)C$.

In multiplication of real numbers there is a multiplicative identity element. Is there an identity element for multiplication of square matrices? If there is, then the product of an $m \times m$ matrix with the identity element will be the $m \times m$ matrix.

The matrix that would satisfy the previous condition is one where all elements are zero except the **principle diagonal**, from upper left to lower right, which has elements of one. The following are examples of identity matrices represented by I .

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 13

Show that $AI = A$ if $A = \begin{bmatrix} 8 & -2 \\ 1 & 4 \end{bmatrix}$ and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution:

$$\begin{aligned} AI &= \begin{bmatrix} 8 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 8(1) + (-2)(0) & 8(0) + (-2)(1) \\ 1(1) + 4(0) & 1(0) + 4(1) \end{bmatrix} \\ &= \begin{bmatrix} 8 & -2 \\ 1 & 4 \end{bmatrix} \end{aligned}$$

Although $AB \neq BA$, $AI = IA = A$.

When addition and multiplication are both involved, you need the **distributive property** to help you perform the operations. Multiplication of matrices is **distributive over addition**. The proof of this property is tedious and will not be shown.

Study the following example which demonstrates the distributive property.

Example 14

Show that $A(B+C) = AB + AC$ if $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$,

$$B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -2 & -1 \\ -1 & -3 \end{bmatrix}.$$

Solution:

$$\begin{aligned} A(B+C) &= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -1 & -3 \end{bmatrix} \right) \\ &= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 0 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} AB + AC &= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 9 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -8 & -9 \\ -3 & -4 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 0 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

Therefore, $A(B+C) = AB + AC$.

Matrix multiplication is associative and distributive if proper attention is paid to the compatibility of the matrices.

There is another strange occurrence in multiplication of matrices. For two matrices A and B , which are not zero matrices, it is possible that their product AB is the zero matrix. The following example shows how this can happen.

Example 15

Show that $AB = 0$ if $A = \begin{bmatrix} 5 & 7 \\ 10 & 14 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 14 \\ 5 & -10 \end{bmatrix}$.

Solution:

$$\begin{aligned} AB &= \begin{bmatrix} 5 & 7 \\ 10 & 14 \end{bmatrix} \begin{bmatrix} -7 & 14 \\ 5 & -10 \end{bmatrix} \\ &= \begin{bmatrix} 5(-7) + 7(5) & 5(14) + 7(-10) \\ 10(-7) + 14(5) & 10(14) + 14(-10) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Neither A nor B is a zero matrix, and yet their product is a zero matrix. (In the set of real numbers, $a \cdot b = 0$ if and only if $a = 0$, $b = 0$, or $a, b = 0$.) A and B are called **divisors of zero**.



From the previous example the set of 2×2 real matrices contain divisors of zero.

Now do questions 4 to 8.

4. Show that $A(BC) = (AB)C$ if $A = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$, and

$$C = \begin{bmatrix} 5 & -3 \\ 0 & 1 \end{bmatrix}.$$

5. Show that $A(B+C) = AB+AC$ if $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$,

$$\text{and } C = \begin{bmatrix} 3 & -3 \\ -2 & 4 \end{bmatrix}.$$

6. Find A^2 if $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$.

7. Find AB if $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 & 1 \\ 1 & 0 & 6 \\ 3 & 4 & 1 \end{bmatrix}$.

8. Find $A^2 + B^2$ if $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$.



For solutions to Activity 2, turn to Appendix A, Topic 1.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

You may decide to do both.



Extra Help

An $m \times n$ real matrix has m rows and n columns. It is said to have dimensions $m \times n$.

For example, $\begin{bmatrix} 3 & 0 \\ 3 & 1 \\ 5 & 6 \end{bmatrix}$ is a 3×2 matrix.

If two matrices have the same dimensions, then they can be added to one another by adding their corresponding elements. Look at the following addition of two matrices.

$$\begin{bmatrix} 3 & 0 & 5 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 3+5 & 0+3 & 5+4 \\ 1+1 & 2+8 & 1+2 \end{bmatrix} = \begin{bmatrix} 8 & 3 & 9 \\ 2 & 10 & 3 \end{bmatrix}$$

It is also possible to subtract two matrices if they have the same dimensions. Subtraction is carried out by subtracting corresponding elements. Look at the following.

$$\begin{bmatrix} 3 & 0 & 5 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 3-5 & 0-3 & 5-4 \\ 1-1 & 2-8 & 1-2 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ 0 & -6 & -1 \end{bmatrix}$$

If you wish to multiply a matrix by a scalar, you must multiply every element by the same scalar.

For example, if $A = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}$, then $2A = \begin{bmatrix} 2(1) & 2(4) \\ 2(3) & 2(7) \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 6 & 14 \end{bmatrix}$.

Matrix multiplication is possible only when the number of columns of the multiplier equals the number of rows of the multiplicand.

An $m \times n$ matrix can be multiplied by an $n \times p$ matrix.

These two numbers must be equal.

The product of two matrices is obtained by multiplying the corresponding elements of the m th row of the first matrix and the n th column of the second matrix and then adding the results. Look at the following example.

Example 16

$$\text{Find } AB \text{ if } A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 3 & -2 & 1 \end{bmatrix}.$$

Solution:

$$\begin{aligned} AB &= \begin{bmatrix} 1(1)+2(0) & 1(2)+2(3) & 1(-1)+2(-2) & 1(4)+2(1) \\ 2(1)+3(0) & 2(2)+3(3) & 2(-1)+3(-2) & 2(4)+3(1) \\ 0(1)+(-1)(0) & 0(2)+(-1)(3) & 0(-1)+(-1)(-2) & 0(4)+(-1)(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 8 & -5 & 6 \\ 2 & 13 & -8 & 11 \\ 0 & -3 & 2 & -1 \end{bmatrix} \end{aligned}$$

Matrix addition is commutative, associative, and closed. Matrix multiplication is associative if proper attention is paid to the compatibility of the matrices, but it is not necessarily commutative. The closure property for multiplication holds for square matrices only.

A square matrix is a matrix with equal number of rows and columns.

A zero matrix (all elements are zeros) is the additive identity in the set of matrices.

A multiplicative identity matrix is a matrix whose principal diagonal from upper left to lower right has elements of 1 while all other elements are zero.

There is a strange occurrence in matrix multiplication. It is possible that the product of two nonzero 2×2 matrices is a zero matrix. Look at the following equation.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

These two matrices are called divisors of zero.

Now do the following questions.

1. Find $A + B$ if $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -2 & 2 \end{bmatrix}$.

2. Find $A - B$ if $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 8 \\ 4 & -9 \end{bmatrix}$.

3. Find $A \times B$ if $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 1 & -1 \\ 3 & 8 & 1 & -2 \end{bmatrix}$.



For solutions to **Extra Help**, turn to **Appendix A, Topic 1**.



Extensions

In matrix multiplication the cancellation law does not hold. Suppose A , B , and C are matrices and $AB = AC$. This does not imply that $B = C$. Look at the following example.

Example 17

If $A = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 7 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & 0 \\ 6 & 9 \end{bmatrix}$, show that $AB = AC$, but $B \neq C$.

Solution:

$$AB = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 7 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ 0 & 0 \end{bmatrix}$$

$AB = AC$, but $B \neq C$.

Now do the following question.

Make up an example which shows that $AB = AC$ where $B \neq C$.



For solutions to **Extensions**, turn to **Appendix A, Topic 1**.

Topic 2 Linear Transformation and Its Inverse



Introduction

A system of equations such as $\begin{cases} u = 2x + y \\ v = 3x - 5y \end{cases}$ represents correspondence (transformation) between two sets of points (u, v) and (x, y) . A transformation can be represented by a matrix of coefficients. Linear transformations, inverse transformations, and the association between matrices and transformations will be discussed in this topic.



What Lies Ahead

Throughout this topic you will learn to

1. define linear transformation, onto transformation, and into transformation
2. determine linear transformations using matrix multiplication
3. determine inverse transformations

Now that you know what to expect, turn the page to begin your study of linear transformations and its inverse.



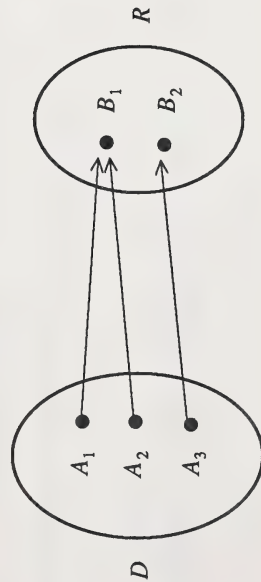
Exploring Topic 2

Activity 1



Define linear transformation, onto transformation, and into transformation.

The following terms are synonymous in mathematics: **function**, **correspondence**, **transformation**, and **mapping**. A function represents the relation between two sets of objects. If D is a set of three points $\{A_1, A_2, A_3\}$, R is a set of two points $\{B_1, B_2\}$, and each element of the set D (domain) is associated in some way with a unique element of the set R (range), then this association is called a function from D to R . The following figure illustrates such an association.



You can say that the function maps D onto R .

Correspondence between sets is very common. For example, to each person there corresponds a name, and to each licence-plate number there corresponds a vehicle.

The equation $y = 2x - 3$ defines a function. For each x -value there corresponds a unique y -value given by this equation. You will find it convenient to use the word *mapping* to describe a particular type of correspondence between two sets. Mapping is represented by the arrows drawn from the points representing the elements of the domain to the points representing the elements of the range.

The system of equations
$$\begin{cases} u = 2x + y \\ v = 3x - 5y \end{cases}$$
 produces the mapping

$(x, y) \rightarrow (u, v)$. In other words, every ordered pair (x, y) is mapped onto another ordered pair (u, v) in the plane represented by all ordered pairs of real numbers.

This mapping or transformation is performed by substituting any real values of x and y into the two equations and calculating the values of u and v . For every point $P(x, y)$ there corresponds a point

$Q(u, v)$. This is called the **image** of P under this mapping.



The following are mappings of $P \rightarrow Q$ from the previous system:

$$(x, y) \rightarrow (u, v)$$

$$(2, 1) \rightarrow (5, 1)$$

$$(0, 0) \rightarrow (0, 0)$$

$$(-2, 3) \rightarrow (-1, -21)$$

[Note: If $(x, y) = (2, 1)$, then

$$u = 2x + y = 5 \text{ and } v = 3x - 5y = 1.]$$

According to the system of equations, when you assign a unique pair of values to x and y , you get a unique pair of values for u and v .

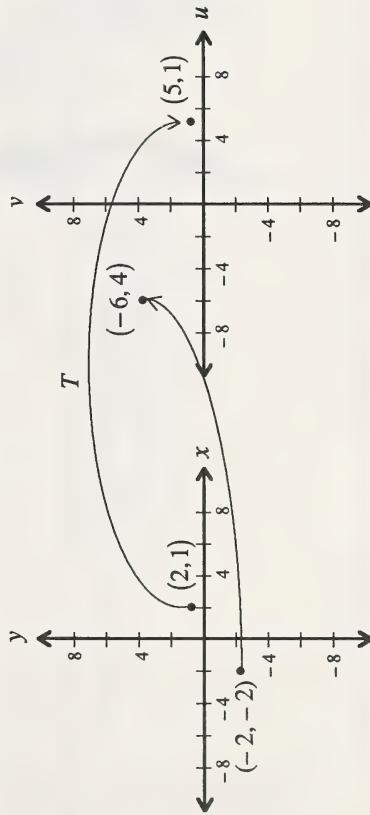
Mappings are classified as either **one-to-one** or **many-to-one**. A mapping is said to be one-to-one if each element of the range is the image under this mapping of exactly one element of the domain. If, however, there is at least one element of the range which is the image under this mapping of two or more elements of the domain, then this mapping is said to be a many-to-one mapping.

Let T be used to represent the mapping or transformation of $T \begin{cases} u = 2x + y \\ v = 3x - 5y \end{cases}$.

You would write $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ to indicate that $\begin{bmatrix} x \\ y \end{bmatrix}$ is mapped onto $\begin{bmatrix} u \\ v \end{bmatrix}$ by the transformation T .

For example, $T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $T \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$.

The following graph illustrates this transformation.



When $x = -2$ and $y = -2$, then
 $u = 2(-2) + (-2) = -6$ and
 $v = 3(-2) - 5(-2) = 4$.

Since you may choose any pair of real numbers for (x, y) , the domain of the mapping is the entire real number $(\mathbb{R} \times \mathbb{R})$ plane. It is apparent that every point P with coordinates (x, y) will have a unique image under this transformation T . Therefore, T is a one-to-one transformation or mapping of the entire plane.

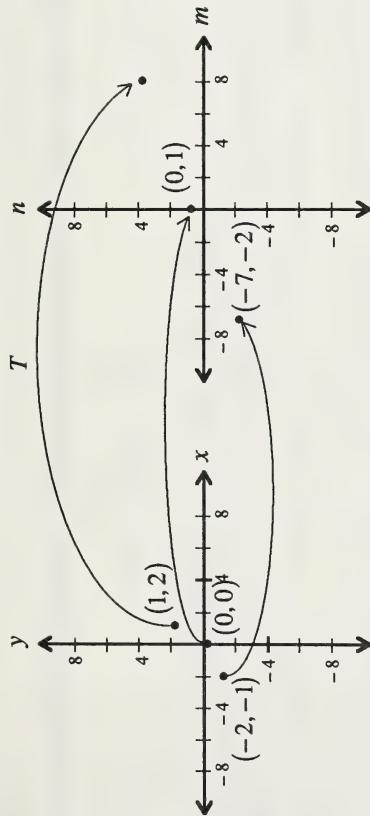


Example 1

Under the transformation $T \begin{cases} m = 2x + 3y \\ n = x + y + 1 \end{cases}$, find the images of the points $(0, 0)$, $(1, 2)$, and $(-2, -1)$, and show the images on a diagram.

Solution:

$$T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \quad T \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \end{bmatrix}$$



It was mentioned that there are two kinds of mappings, namely one-to-one and many-to-one. If a classroom is full of students seated in chairs, then there exists a one-to-one mapping of the set of students **onto** the set of chairs in the room.

Consider the transformation $T \begin{cases} u = 2x + y \\ v = 3x - 5y \end{cases}$ again. You already know that each point with

coordinates (x, y) corresponds to another unique point with coordinates (u, v) . To prove that it is unique, prove that each point with coordinates (u, v) corresponds to another unique point with coordinates (x, y) . Suppose you try to use u and v to represent x and y .

$$u = 2x + y \quad (1)$$

$$v = 3x - 5y \quad (2)$$

$$5 \times (1) + (2): 5u + v = 13x$$

$$x = \frac{5}{13}u + \frac{v}{13}$$

$$\text{At } x = 0 \text{ and } y = 0,$$

$$m = 2(0) + 3(0)$$

$$= 0$$

$$n = 0 + 0 + 1$$

$$= 1$$

$$\text{At } x = 1 \text{ and } y = 2,$$

$$m = 2(1) + 3(2)$$

$$= 8$$

$$n = 1 + 2 + 1$$

$$= 4$$

$$\text{At } x = -2 \text{ and } y = -1,$$

$$m = 2(-2) + 3(-1)$$

$$= -7$$

$$n = (-2) + (-1) + 1$$

$$= -2$$

$$3 \times (1) - 2 \times (2): 3u - 2v = 13y$$

$$y = \frac{3}{13}u - \frac{2}{13}v$$

Thus, $\begin{cases} u = 2x + y \\ v = 3x - 5y \end{cases}$ is equivalent to $\begin{cases} x = \frac{5}{13}u + \frac{1}{13}v \\ y = \frac{2}{13}u - \frac{2}{13}v \end{cases}$.



From this conclusion you can see that there is a unique point (x, y) for every point (u, v) . Such a one-to-one mapping is called **onto mapping**.

Is there another kind of mapping? Yes. It is called **into mapping**. Consider the following transformation.

$$T \begin{cases} u = x + 2y \\ v = 2x + 4y \end{cases}$$

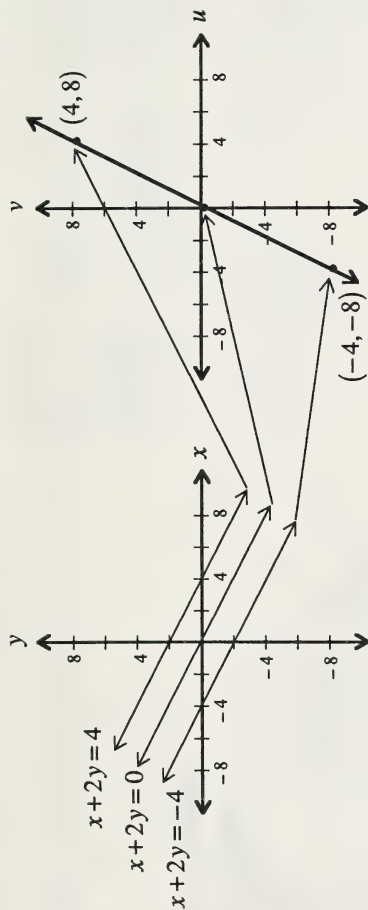
Under this transformation every point in the xy -plane will have an image in the uv -plane. Since $v = 2u$, the image points form a straight line with equation $v = 2u$. This means that the other points which are not on the line with equation $v = 2u$ are not image points, and the xy -plane and uv -plane do not have a one-to-one correspondence. This is a transformation of the xy -plane into the uv -plane.



In the following diagram $x + 2y = k$, $k \in R$ represents a family of parallel lines, and $(k, 2k)$ represents the image of each parallel line. The images form a straight line. The equation of this straight line is $v = 2u$.

$$\begin{aligned} v &= 2x + 4y \\ &= 2(x + 2y) \\ &= 2u \end{aligned}$$

For the line $x + 2y = -4$, when $x = 0$, then $y = -2$. Substitute $x = 0$ and $y = -2$ into $u = x + 2y$; thus, $u = 0 + 2(-2) = -4$. Substitute $x = 0$ and $y = -2$ into $v = 2x + 4y$; thus, $v = 2(0) + 4(-2) = -8$. Other values of x and y satisfying $x + 2y = -4$ will result in $u = -4$ and $v = -8$. Therefore, all values on the line $x + 2y = -4$ map to the point $(-4, -8)$. This is many-to-one mapping.



This is a many-to-one mapping. It is a transformation of the xy -plane into the uv -plane, or a transformation of the xy -plane onto the line $v = 2u$.

Look at the following example.

Example 2

Is $T \begin{cases} x = 3u - v \\ y = u + 2v \end{cases}$ an onto or into mapping?

Solution:

T produces an image for every point (u, v) . Every point in the xy -plane corresponds to a unique point in the uv -plane and vice versa. It is a one-to-one correspondence and is an onto mapping.

Example 3

Is $S \begin{cases} u = 2x - y \\ v = 3x - \frac{3}{2}y \end{cases}$ an onto or into mapping?

Solution:

Since $u = \frac{2}{3}v$, every point in the xy -plane can be mapped onto the straight line $u = \frac{2}{3}v$. Since the other points not on the line with equation $u = \frac{2}{3}v$ are not image points, it is a transformation of the xy -plane into the uv -plane or a transformation of the xy -plane onto the line $u = \frac{2}{3}v$.



1. For the transformation $T \begin{cases} u = x - 3y \\ v = 2x + 5y \end{cases}$, find the images of the following points by using the substitution method, and show these points and their images on a graph. Graph paper is provided in Appendix B.

a. $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

b. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

d. $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$

2. Find $V \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ for the transformation $V \begin{cases} R = 2r \\ S = -3s \end{cases}$.

3. For the transformation $y = x^2$, determine whether the following mappings are into or onto mappings.

- a. Set A for x Set B for y



b. Set A for x

Set B for y



4. Find (x, y) if $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $T \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3x - 2y \\ x + y \end{bmatrix}$.

5. Explain how $S \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} r - 2s \\ 3r - s \end{bmatrix}$ is a one-to-one mapping. Is this an onto or into mapping?

6. Find $S \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for the transformation $S \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3x - 5y \\ x + 3y \end{bmatrix}$.

7. Explain how the transformation $T \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x - \frac{1}{4}y \\ \frac{1}{6}x - \frac{1}{8}y \end{bmatrix}$ is a mapping of the xy -plane into the uv -plane.



For solutions to **Activity 1**, turn to **Appendix A, Topic 2**.

Activity 2



Determine linear transformations using matrix multiplication.

The basic ingredients of a transformation are the coefficients of the variables in the system of equations. The following transformations are all essentially the same transformation.

$$\begin{cases} u = 2x - 5y \\ v = x + 3y \end{cases} \quad \begin{cases} \lambda = 2\alpha - 5\beta \\ \sigma = \alpha + 3\beta \end{cases} \quad \begin{cases} m = 2r - 5s \\ n = r + 3s \end{cases}$$

As you can see, the previous transformation is determined by the coefficients 2, -5, 1, and 3. Thus, you may now associate a matrix with every transformation.

For example, $T \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5x - 3y \\ x + 6y \end{bmatrix}$ corresponds to the matrix of coefficients $T = \begin{bmatrix} 5 & -3 \\ 1 & 6 \end{bmatrix}$. The same symbol T can be used to refer

to both the transformation and the matrix corresponding to the transformation.

Look at the following examples.

Example 4

Write the matrix associated with the transformation $R \begin{cases} u = x + 2y \\ v = 3x - y \end{cases}$.

Solution:

The corresponding matrix of coefficients is $R = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$.

Example 5

Write a linear transformation corresponding to the matrix

$$T = \begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix}.$$

Solution:

$$T \begin{cases} u = -2x + 5y \\ v = 3x - 7y \end{cases}$$

It was mentioned earlier that you could write $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ given

the transformation $T \begin{cases} u = -2x + 5y \\ v = 3x - 7y \end{cases}$. This indicates that $\begin{bmatrix} x \\ y \end{bmatrix}$ can

be mapped on $\begin{bmatrix} u \\ v \end{bmatrix}$ by means of this transformation. If T is

replaced by the corresponding matrix of coefficients, then

$$\begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

This becomes a matrix equation. The matrix $\begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix}$

indicates that an operation when performed on the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$

results in the elements changing to $\begin{bmatrix} u \\ v \end{bmatrix}$.

What is this operation? How does it work?

Substitute $(2, 1)$ for (x, y) into the system of equations.

$$u = -2(2) + 5(1) = 1$$

$$v = 3(2) - 7(1) = -1$$

$$\text{In other words, } \begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2(2) + 5(1) \\ 3(2) - 7(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since $\begin{bmatrix} -2(2)+5(1) \\ 3(2)-7(1) \end{bmatrix}$ is the product of the

two matrices $\begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, the

operation $\begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a matrix

multiplication. Now you know that the image of a given point under a given transformation can be found by using matrix multiplication.



Example 6

Find $T \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ if T is the transformation

corresponding to the matrix $\begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix}$.

Solution:

$$\begin{aligned} T \begin{bmatrix} -1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3(-1)+0(2) \\ 1(-1)+4(2) \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 7 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2(2)+5(1) \\ 3(2)-7(1) \end{bmatrix}$$

$$\begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2(2)+5(1) \\ 3(2)-7(1) \end{bmatrix}$$

$(-3, 7)$ is called the image of point $(-1, 2)$.

Example 7

Show that the images P'_1 , P'_2 , and P'_3 of P_1 , P_2 , and P_3 respectively are collinear under the

transformation $S \begin{cases} u = x + 2y \\ v = x - 3y \end{cases}$ given three

collinear points $P_1(1, -2)$, $P_2(3, 2)$, and $P_3(5, 6)$.

Solution:

$$\begin{aligned}
 S \begin{bmatrix} 1 \\ -2 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
 &= \begin{bmatrix} 1(1)+2(-2) \\ 1(1)+(-3)(-2) \end{bmatrix} \\
 &= \begin{bmatrix} -3 \\ 7 \end{bmatrix} \\
 &= P_1'
 \end{aligned}$$

$$\begin{aligned}
 S \begin{bmatrix} 3 \\ 2 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} 1(3)+2(2) \\ 1(3)+(-3)2 \end{bmatrix} \\
 &= \begin{bmatrix} 7 \\ -3 \end{bmatrix} \\
 &= P_2'
 \end{aligned}$$

$$\begin{aligned}
 S \begin{bmatrix} 5 \\ 6 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} \\
 &= \begin{bmatrix} 1(5)+2(6) \\ 1(5)+(-3)(6) \end{bmatrix} \\
 &= \begin{bmatrix} 17 \\ -13 \end{bmatrix} \\
 &= P_3'
 \end{aligned}$$

$$\begin{aligned}
 \text{Slope of } P_1' P_2' &= \frac{-3-7}{7-(-3)} \\
 &= \frac{-10}{10} \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{Slope of } P_2' P_3' &= \frac{-3+13}{7-17} \\
 &= \frac{+10}{-10} \\
 &= -1
 \end{aligned}$$

Therefore, P_1' , P_2' , and P_3' are collinear.

Now do at least the first two questions.

- Find the set of points whose image is $W \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

for the transformation $W \begin{cases} u = x - 3y \\ v = 2x - 6y \end{cases}$.

- Find (r, s) under $W \begin{cases} m = -2r - s \\ n = 3s \end{cases}$ if

$$W \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

- State the matrices corresponding to the following transformations.

- $T \begin{cases} u = -x - 5y \\ v = 3x + 7y \end{cases}$
- $S \begin{cases} m = 2r - s \\ n = -r \end{cases}$

- State the transformations corresponding to the following matrices.

- $\begin{bmatrix} 3 & -2 \\ 0 & 5 \end{bmatrix}$
- $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

- Find the images of the following points by using matrix multiplication for the

transformation $T \begin{cases} u = x - 3y \\ v = 2x + 5y \end{cases}$.

- $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

- $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

- $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$



For solutions to Activity 2, turn to Appendix A, Topic 2.



Activity 3



Determine inverse transformations.

The symbol f is used to denote a function. Its inverse is frequently designated by f^{-1} . Since a function is a transformation, the same notation is used to designate transformations. If T is a one-to-one transformation, then the **inverse transformation** T^{-1} should exist and T^{-1} would cancel what the transformation has done. The inverse operation of adding 2 is the operation of subtracting 2. If the transformation T maps $(1, 3)$ onto $(4, 5)$, then the inverse transformation T^{-1} must map $(4, 5)$ back onto $(1, 3)$. If T is a one-to-one transformation of the xy -plane onto the uv -plane, then T^{-1} is a transformation of the uv -plane onto the xy -plane.

Now study the following example.

Example 8

- For $T \begin{cases} u = 2x - y \\ v = x + 3y \end{cases}$ find T^{-1} by solving the system of equations.

Solution:

$$\begin{aligned} 2x - y &= u & (1) \\ x + 3y &= v & (2) \end{aligned}$$

$$\begin{aligned} 3 \times (1) + (2): \quad 7x &= 3u + v \\ x &= \frac{3}{7}u + \frac{1}{7}v \\ -2 \times (1) - (2): \quad -7y &= u - 2v \\ y &= \frac{2}{7}v - \frac{1}{7}u \end{aligned}$$

$$\therefore T^{-1} \begin{cases} x = \frac{3}{7}u + \frac{1}{7}v \\ y = \frac{2}{7}v - \frac{1}{7}u \end{cases}$$

This is a transformation of the uv -plane onto the xy -plane.

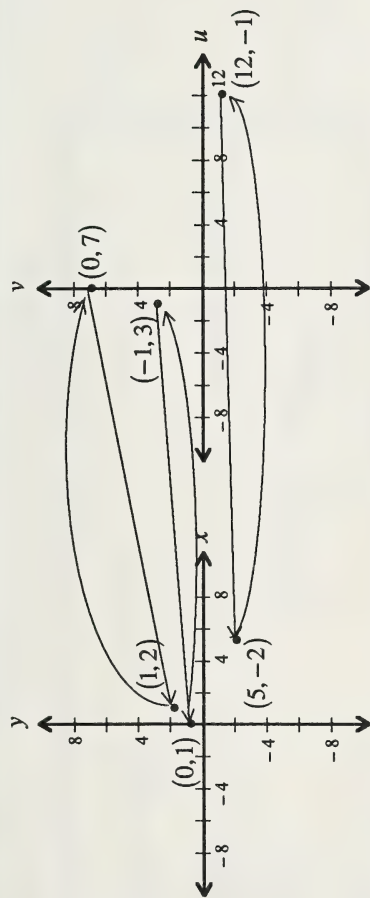
- Choose some specific points to illustrate how T^{-1} cancels T .

Solution:

$$\text{For } (x, y) = (1, 2), T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix} \text{ and } T^{-1} \begin{bmatrix} 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$\text{For } (x, y) = (5, -2), T \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \end{bmatrix} \text{ and } T^{-1} \begin{bmatrix} 12 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

$$\text{For } (x, y) = (0, 1), T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \text{ and } T^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



The previous example is a one-to-one transformation. Consider another transformation which is not a one-to-one transformation.

$$S \begin{cases} u = 2x - y \\ v = 4x - 2y \end{cases}$$

$(4x - 2y) = 2(2x - y)$ implies that $v = 2u$. The xy -plane is covered by the family of parallel lines $2x - y = k$, $k \in R$. The image of each of these parallel lines is a point $(k, 2k)$ on the line $v = 2u$ in the uv -plane. This transformation does not produce a one-to-one correspondence between the points of the two planes. For instance, $(0, -1)$, $(1, 1)$, and $(2, 3)$ of the xy -plane all map onto the single point $(1, 2)$ in the uv -plane. You cannot reverse this mapping because you do not know whether to map $(1, 2)$ back to $(0, -1)$ or $(2, 3)$. There can be no inverse for this transformation. This transformation is a many-to-one transformation.

Therefore, the transformation $T \begin{cases} u = ax + by \\ v = cx + dy \end{cases}$ will have an inverse T^{-1} if and only if T is a one-to-one

mapping of the xy -plane onto the uv -plane. An into transformation does not have an inverse. However, this conclusion is the result of inductive reasoning which is not a proof.

The general expression of T^{-1} can be derived as follows.

Consider the following general expression:

$$T \begin{cases} u = ax + by & (1) \\ v = cx + dy & (2) \end{cases}$$

T^{-1} is found by solving this system of equations for x and y . If you have difficulty solving an equation system with literal coefficients, then follow this solution.

$$d \times (1): \quad adx + bdy = du \quad (3)$$

$$b \times (2): \quad bcx + bdy = bv \quad (4)$$

$$(3) - (4): \quad adx - bcx = du - bv$$

$$x = \frac{du - bv}{ad - bc}$$

$$= \frac{d}{ad - bc} u - \frac{b}{ad - bc} v$$

$$c \times (1): \quad acx + bcy = cu \quad (5)$$

$$a \times (2): \quad acx + ady = av \quad (6)$$

$$(5) - (6): \quad (bc - ad)y = cu - av$$

$$y = \frac{cu - av}{bc - ad}$$

$$= \frac{-cu + av}{ad - bc}$$

$$= \frac{-c}{ad - bc} u + \frac{a}{ad - bc} v$$

$$\therefore T^{-1} \begin{cases} x = \frac{d}{ad - bc} u - \frac{b}{ad - bc} v \\ y = \frac{-c}{ad - bc} u + \frac{a}{ad - bc} v \end{cases}$$

Since the denominator is $ad - bc$, T^{-1} exists if and only if $ad - bc \neq 0$. This general expression of T^{-1} is very useful.

Study the following examples.

Example 9

Determine whether there is an inverse

transformation for $W \begin{cases} u = 2x - y \\ v = 3x + 2y \end{cases}$.

Solution:

$$\begin{aligned} ad - bc &= (2)(2) - (-1)(3) \\ &= 7 \quad (\text{does not equal zero}) \end{aligned}$$

Therefore, W^{-1} exists.

For $T \begin{cases} u = ax + by \\ v = cx + dy \end{cases}$, an inverse transformation exists if and only if $ad - bc \neq 0$.

Example 10

For the transformation $T \begin{cases} u = 3x \\ v = -2x - y \end{cases}$ find the

point (x, y) such that $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution:

$$\begin{aligned} x &= \frac{du - bv}{ad - bc} \\ &= \frac{(-1)(0) - 0(0)}{3(-1) - 0(-2)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} y &= \frac{-cu + av}{ad - bc} \\ &= \frac{-(-2)(0) + 3(0)}{3(-1) - 0(-2)} \\ &= 0 \end{aligned}$$

Therefore, the point (x, y) is $(0, 0)$.

Example 11

Find the point (x, y) such that $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ for

the transformation $T \begin{cases} u = 3x \\ v = -2x - y \end{cases}$.

Solution:

$$\begin{aligned} x &= \frac{(-1)(-2) - 0(-3)}{3(-1) - 0(-2)} \\ &= -\frac{2}{3} \end{aligned}$$

$$\begin{aligned} y &= \frac{-(-2)(-2) + 3(-3)}{3(-1) - 0(-2)} \\ &= \frac{13}{3} \end{aligned}$$

Therefore, the point (x, y) is $(-\frac{2}{3}, \frac{13}{3})$.

If you do not want to use the formulas, you can solve each system of equations for x and y . If you want to use the formulas, you will have to memorize them.

Now do the following exercises.

1. Find the inverse of each of the following transformations if they exist.

a. $T \begin{cases} u = -5x + y \\ v = x - y \end{cases}$

b. $A \begin{cases} u = -2x \\ v = 5y \end{cases}$

c. $W \begin{cases} u = 3x - 2y \\ v = 0 \end{cases}$

d. $S \begin{cases} u = -x + 2y \\ v = 2x - 4y \end{cases}$

2. Find the point (r, s) if its image is $(-1, 2)$ for the transformation $B \begin{cases} m = r + 3s \\ n = -2r + s \end{cases}$.

3. For $T \begin{cases} u = -x - 3y \\ v = x - y \end{cases}$ find (x, y) if the image of the point (x, y) is as follows.

a. $(-2, -3)$

b. $(0, 0)$



For solutions to **Activity 3**, turn to **Appendix A, Topic 2**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

You may decide to do both.



Extra Help

A system of linear equations like $\begin{cases} u = x - 2y \\ v = 2x + y \end{cases}$ represents the relation

between two sets of points (x, y) and (u, v) . By means of these equations, every ordered pair (x, y) in the xy -plane can be mapped onto another ordered pair (u, v) in the uv -plane. This corresponding point (u, v) is called the image of (x, y) . This type of correspondence between two sets is called linear transformation or mapping.

For example, if $(x, y) = (2, 1)$, then the image (u, v) can be solved by substituting into the original system.

$$\begin{aligned} (u, v) &= (2 - 2(1), 2(2) + 1) \\ &= (0, 5) \end{aligned}$$

If each point of the uv -plane is the image of at least one point of the xy -plane, then this transformation is a mapping of the xy -plane onto the uv -plane. If some points in the uv -plane are not images of points in the xy -plane, then the transformation is a mapping of the xy -plane into the uv -plane.

For example, $T \begin{cases} u = x - y \\ v = 2x - 2y \end{cases}$ is a transformation under which every

point in the xy -plane will have an image in the line $2u = v$. Those points not on the line $2u = v$ are not images of any point in the xy -plane. This transformation is a mapping of the xy -plane into the uv -plane or onto the line $2u = v$.

Matrix multiplication can be used to perform linear transformations.

For example, $T \begin{cases} u = 2x - y \\ v = 3x + 5y \end{cases}$ can be replaced by $\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$.

$$\begin{aligned} \text{If } (x, y) &= (3, 4), \text{ then } \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 2(3) + (-1)4 \\ 3(3) + 5(4) \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 29 \end{bmatrix}. \end{aligned}$$

Therefore, $(u, v) = (2, 29)$.

If T is a one-to-one transformation of the xy -plane onto the uv -plane, it is possible to have a transformation of the uv -plane onto the xy -plane. The transformation is called the inverse transformation of T . The inverse transformation of T is denoted by T^{-1} . T^{-1} can be obtained by solving the system of equations for x and y or by using the following formula.

$$\text{If } T \begin{cases} u = ax + by \\ v = cx + dy \end{cases}, \text{ then } T^{-1} \begin{cases} x = \frac{d}{ad-bc}u - \frac{b}{ad-bc}v \\ y = \frac{-c}{ad-bc}u + \frac{a}{ad-bc}v \end{cases}$$

The following example shows you how to use this formula.



Example 12

Find T^{-1} for $T \begin{cases} u = x + 2y \\ v = 3x - 2y \end{cases}$.

Solution:

$$a = 1, b = 2, c = 3, d = -2$$

$$T^{-1} \begin{cases} x = \frac{(-2)}{(1)(-2) - (2)(3)}u - \frac{2}{(1)(-2) - (2)(3)}v \\ y = \frac{-3}{(1)(-2) - (2)(3)}u + \frac{1}{(1)(-2) - (2)(3)}v \end{cases}$$

$$T^{-1} \begin{cases} x = \frac{2}{8}u + \frac{2}{8}v \\ y = \frac{3}{8}u - \frac{1}{8}v \end{cases}$$

$$T^{-1} \begin{cases} x = \frac{1}{4}u + \frac{1}{4}v \\ y = \frac{3}{8}u - \frac{1}{8}v \end{cases}$$

Now do the following questions.

- Find the point (u, v) such that $T \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

if $T \begin{cases} u = x - 2y \\ v = 5x + y \end{cases}$ using the following.

- the substitution method
- matrix multiplication

- Find the point (x, y) such that $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

if $T \begin{cases} u = x - 2y \\ v = 5x + y \end{cases}$.

- Find T^{-1} if $T \begin{cases} u = 2y \\ v = x - y \end{cases}$.



For solutions to Extra Help, turn to Appendix A, Topic 2.



Extensions

There are many theorems which tell you the properties of an inverse matrix. They are beyond the scope of this course. If you are looking for more challenging explorations, do the following question. This question represents a special case of the theorem

$(AB)^{-1} = B^{-1}A^{-1}$, where A and B are matrices of the same order and not singular.

Prove that $(AB)^{-1} = B^{-1}A^{-1}$ if $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ and

$$B = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}.$$



For solutions to Extensions, turn to Appendix A, Topic 2.

If a matrix has no inverse, it is called a **singular matrix**.

Topic 3 Product of Two Transformations



Introduction

You have learned about multiplication of matrices. Since every transformation corresponds to a matrix, multiplication of two transformations is multiplication of two matrices. In this topic you are going to learn what the product of two transformations means.



What Lies Ahead

Throughout this topic you will learn to

1. define the product of two transformations

Now that you know what to expect, turn the page to begin your study of the product of two transformations.



Exploring Topic 3

Activity 1



Define the product of two transformations.

Suppose you have a point (x, y) in the xy -plane and it is mapped onto another point (u, v) in the uv -plane by a transformation T . Follow this by a second transformation S which maps the point (u, v) onto a third point (r, s) in the rs -plane.

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \quad S \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$$

The performance of these two transformations, T followed by S , has the total effect of mapping every ordered pair (x, y) onto an ordered pair

(r, s) . You can use $ST \begin{bmatrix} x \\ y \end{bmatrix}$ to indicate this

compound operation to be performed on (x, y) .

The combination of two transformations, T and S , is called the product ST .

The product ST is a transformation which can be obtained by carrying out the first transformation T and following this with the second transformation S . ST can also be obtained by multiplying the two matrices which represent the two transformations T and S .

$$ST \begin{bmatrix} x \\ y \end{bmatrix} = S \left(T \begin{bmatrix} x \\ y \end{bmatrix} \right) = S \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$$

Note the apparent reversal of the normal order.

The following example shows you three different ways to find the product of two transformations.

Example 1

If $T \begin{cases} u = 2x - 3y \\ v = x - 3y \end{cases}$ and $S \begin{cases} m = 3u - v \\ n = u + v \end{cases}$, find $ST \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Note that it is ST , not TS .

Transformation T comes first.

Solution:

$$ST \begin{bmatrix} 2 \\ 1 \end{bmatrix} = S \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$$

$$T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(2) + (-3)(1) \\ 1(2) + (-3)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore S \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = S \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1) + (-1)(-1) \\ 1 + (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

You may want to follow the procedure for matrix multiplication and present the solution in this manner.

$$T = \begin{bmatrix} 2 & -3 \\ 1 & -3 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore ST \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2(2) + (-3)(1) \\ 1(2) + (-3)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1) + (-1)(-1) \\ 1(1) + 1(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

You may present the solution by simply substituting (x, y) for (u, v) .

$$\begin{aligned}
 m &= 3u - v \\
 &= 3(2x - 3y) - (x - 3y) \\
 &= 5x - 6y \\
 n &= u + v \\
 &= (2x - 3y) + (x - 3y) \\
 &= 3x - 6y
 \end{aligned}$$

When $x = 2$, then $y = 1$.

$$\begin{aligned}
 m &= 5(2) - 6(1) \\
 &= 4 \\
 n &= 3(2) - 6(1) \\
 &= 0
 \end{aligned}$$

Therefore, $ST \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.



Since multiplication of matrices is **not** commutative,

$$ST \neq TS \text{ and } T \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} x \\ y \end{bmatrix} T.$$

Try the following questions.

- For the transformations $T \begin{cases} u = 2x - 3y \\ v = x - 3y \end{cases}$ and $S \begin{cases} m = 3u - v \\ n = u + v \end{cases}$, find each of the following by multiplying the transformations.

- $ST \begin{bmatrix} -2 \\ 3 \end{bmatrix}$
- $ST \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

- For the transformations $A \begin{cases} u = x - 4y \\ v = 2x - y \end{cases}$ and $B \begin{cases} m = u - 2v \\ n = -u + v \end{cases}$, find $A \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $B \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, and $B \left(A \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right)$.



For solutions to Activity 1, turn to **Appendix A, Topic 3**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

You may decide to do both.



Extra Help

The product of two transformations is the product of two matrices. It has the effect of two mappings. It maps a point from the first plane to the second plane and then to the third plane.

If T is the first transformation and S is the second transformation,

then this compound operation on (x, y) should be written as $ST \begin{bmatrix} x \\ y \end{bmatrix}$.

If $T \begin{cases} u = x + 3y \\ v = 2x - y \end{cases}$ and $S \begin{cases} m = 2u + v \\ n = -u - v \end{cases}$, then $ST \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ would map the point

$(1, 2)$ onto (m, n) . To determine (m, n) , you have to multiply the following matrices.

$$\begin{aligned} ST \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 14 \\ -7 \end{bmatrix} \end{aligned}$$

Therefore, $(m, n) = (14, -7)$.

Now do the following questions.

1. For the transformations $A \begin{cases} u = 0x + y \\ v = 2x + 3y \end{cases}$ and $B \begin{cases} r = 3u - 2v \\ s = u + v \end{cases}$, find

$$BA \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

2. For the transformations $A \begin{cases} u = x - y \\ v = 3x - 2y \end{cases}$ and $B \begin{cases} m = 3u - v \\ n = 2v \end{cases}$, find

$$BA \begin{bmatrix} -3 \\ 0 \end{bmatrix}.$$



For solutions to Extra Help, turn to Appendix A, Topic 3.



Extensions

This result indicates that the image of the point $(x, y) = (-2, 1)$ is $(r, s) = (-206, 43)$.

Now try the following question.

For the transformations $A \begin{cases} x = 3u - v \\ y = 5u - v \end{cases}, B \begin{cases} u = r + 3s \\ v = 3r \end{cases}$, and

$$C \begin{cases} r = 2n \\ s = m - n \end{cases}, \text{ find } ABC \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Example 2

For the transformations $A \begin{cases} u = 8x - y \\ v = 6x - y \end{cases}, B \begin{cases} m = u + v \\ n = 3u + 4v \end{cases}$, and

$$C \begin{cases} r = 2n \\ s = 2m - n \end{cases}, \text{ find } CBA \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Solution:

$$\begin{aligned} CBA \begin{bmatrix} -2 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 8 & -1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -17 \\ -13 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -30 \\ -103 \end{bmatrix} \\ &= \begin{bmatrix} -206 \\ 43 \end{bmatrix} \end{aligned}$$



For the solution to Extensions, turn to **Appendix A, Topic 3**.



Topic 4 Special Transformations



Introduction

In this topic you will look at some special linear transformations. Rotation, magnification, and reflection will be some of these transformations. Each transformation will have a special effect on the image. For example, a shear can transform a rectangle into a parallelogram. There are many of them and you will have to remember the characteristics of each of them.



What Lies Ahead

Throughout this topic you will learn to

1. explain transformations dealing with the identity transformation, the zero transformation, rotations, magnifications, projections, reflections, and shears

Now that you know what to expect, turn the page to begin your study of special transformations.



Exploring Topic 4

Activity 1



Explain transformations dealing with the identity transformation, the zero transformation, rotations, magnifications, projections, reflections, and shears.

Recall that the identity matrix I is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The

product of a matrix and an identity matrix is equal to the original matrix. Therefore, under an **identity transformation** (designated by an identity matrix), the image of a point (x, y) will be identical to the

original point. In other words, $I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.



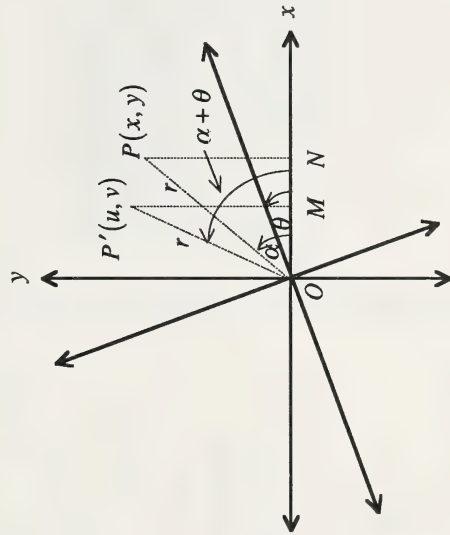
Recall that the zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and under a **zero transformation** designated by a zero matrix, the image of a point (x, y) is always $(0, 0)$.



Consider a **rotation** in which every point rotates θ° about the origin $(0, 0)$. This actually is a mapping of the whole plane onto itself and it is a one-to-one mapping, but its orientation will be altered.

If rotation is involved, you need trigonometric functions to do the job. It is not easy to determine such a transformation. The method used to determine such a transformation will be shown. You do not have to remember the procedure, but you have to remember the conclusion.

Let P be any point (x, y) as shown in the following diagram.



If the plane rotates through $\angle \theta$, then $P(x, y)$ moves to $P'(u, v)$.

Let r denote the distance between the origin and the point P , and let α denote the angle from the x -axis to the line OP . The length of the OP' is also r . The angle from the x -axis to the line OP' is $(\alpha + \theta)$.

In $\triangle OPN$, $x = r \cos \alpha$ and $y = r \sin \alpha$.

In $\triangle OP'M$, $u = r \cos(\alpha + \theta)$ and $v = r \sin(\alpha + \theta)$.

The basic formulas for the cosine and sine of the sum of two angles are as follows:

$$\cos(\alpha + \theta) = \cos \alpha \cos \theta - \sin \alpha \sin \theta$$

$$\sin(\alpha + \theta) = \sin \alpha \cos \theta + \cos \alpha \sin \theta$$

Now use the sine and cosine formulas to solve for u and v .

$$\begin{aligned} u &= r[\cos \alpha \cos \theta - \sin \alpha \sin \theta] \\ &= r \cos \alpha \cos \theta - r \sin \alpha \sin \theta \\ &= x \cos \theta - y \sin \theta \end{aligned}$$

$$\begin{aligned} v &= r[\sin \alpha \cos \theta + \cos \alpha \sin \theta] \\ &= r \sin \alpha \cos \theta + r \cos \alpha \sin \theta \\ &= y \cos \theta + x \sin \theta \end{aligned}$$

Therefore, the system of equations is as follows:

$$u = x \cos \theta - y \sin \theta \quad (1)$$

$$v = x \sin \theta + y \cos \theta \quad (2)$$

If you use R_θ to denote this transformation, then the corresponding matrix is

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$



Here, R_θ represents a counterclockwise rotation in the plane through angle θ .

For this matrix, $a = \cos \theta$, $b = -\sin \theta$,

$c = \sin \theta$, $d = \cos \theta$, and

$$ad - bc = \cos^2 \theta + \sin^2 \theta = 1.$$

An inverse transformation and matrix exist.

They can be determined by solving the system of equations for x and y in terms of u and v .

Now use the following system of equations to solve for x .

$$\begin{aligned} u &= x \cos \theta - y \sin \theta & \textcircled{1} \\ v &= x \sin \theta + y \cos \theta & \textcircled{2} \end{aligned}$$

$$\begin{aligned} \cos \theta \times \textcircled{1}: & \quad u \cos \theta = x \cos^2 \theta - y \sin \theta \cos \theta & \textcircled{3} \\ \sin \theta \times \textcircled{2}: & \quad v \sin \theta = x \sin^2 \theta + y \sin \theta \cos \theta & \textcircled{4} \\ \textcircled{3} + \textcircled{4}: & \quad u \cos \theta + v \sin \theta = x \cos^2 \theta + x \sin^2 \theta \\ & \quad = x(\cos^2 \theta + \sin^2 \theta) \\ & \quad = x \end{aligned}$$

Therefore, $x = u \cos \theta + v \sin \theta$.

By a similar method you can find $y = -u \sin \theta + v \cos \theta$.

Therefore, $\begin{cases} x = u \cos \theta + v \sin \theta \\ y = -u \sin \theta + v \cos \theta \end{cases}$ is the inverse transformation (clockwise rotation).



It represents a rotation through $\angle -\theta$ because the inverse matrix is

$$R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.$$

Now use the previous two formulas to solve some problems.

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\begin{aligned} \cos \theta &= \cos(-\theta) \\ -\sin \theta &= \sin(-\theta) \end{aligned}$$

Example 1

The points $(1, 2)$ and $(-1, 1)$ are rotated counterclockwise about the origin through 45° . Find their images and show the points and their images on the same diagram.

Solution:

$$R_{\frac{\pi}{4}} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$R_{\frac{\pi}{4}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix}$$

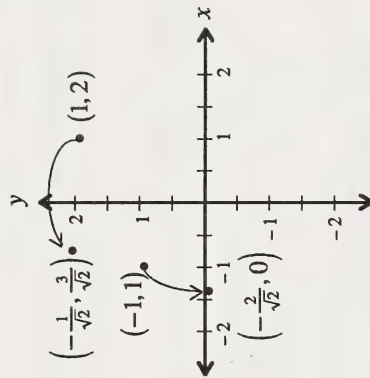
Therefore, the image of $(1, 2)$ is

$$\left(\frac{-1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) \doteq (-0.707, 2.12).$$

$$R_{\frac{\pi}{4}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Therefore, the image of $(-1, 1)$ is

$$\left(\frac{-2}{\sqrt{2}}, 0\right) \doteq (-1.414, 0).$$



The next example is one of clockwise rotation.

Example 2

The point $(-1, 1)$ is rotated about the origin through -45° . Find the image.

Solution:

$$R_{-\frac{\pi}{4}} = \begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$R_{-\frac{\pi}{4}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{2}} \end{bmatrix}$$

Therefore, the image of $(-1, 1)$ is $\left(0, \frac{2}{\sqrt{2}}\right)$.

$$R_\theta = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

The image is on the y -axis.

Uniform expansion or contraction can make the image larger or smaller than the original figure. A

transformation of this kind is called **uniform magnification**.

Under the transformation

$$M \begin{cases} u = kx + 0y \\ v = 0x + ky \end{cases} \text{ or } M = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix},$$

both x and y would be magnified k times. The transformation is an expansion if $k > 1$ and is a contraction if $0 < k < 1$.

See how this transformation works.

Example 3

Find the image of the segment determined by $P(1, 1)$ and $Q(3, 4)$ under the transformation

$$M \begin{cases} u = 2x \\ v = 2y \end{cases}.$$

Solution:

$$\text{For } M \begin{cases} u = 2x \\ v = 2y \end{cases}, k = 2 \text{ and } M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$



M represents the magnification transformation.

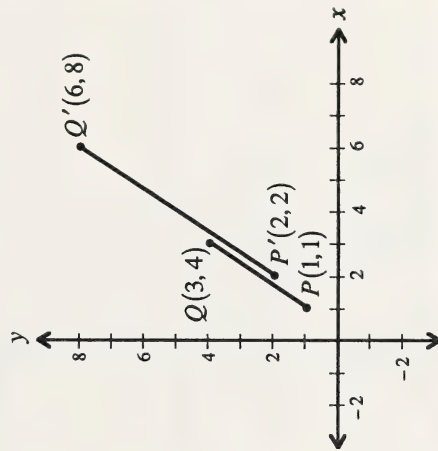
$$M \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$M \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Therefore, the image is segment $P'Q'$ where P' is $(2, 2)$ and Q' is $(6, 8)$. The length of $P'Q'$ is two times the length of PQ .



$$PQ = \sqrt{(3-1)^2 + (4-1)^2}$$

$$= \sqrt{13}$$

$$P'Q' = \sqrt{(6-2)^2 + (8-2)^2}$$

$$= \sqrt{52}$$

$$= 2\sqrt{13}$$

If you change the transformation

to $M_x \begin{cases} u = kx + 0y \\ v = 0x + 1y \end{cases}$ or

$M_x = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, then the

expansion is along the x -axis only. You can call it the magnification in the x -direction.



Example 4

Find the image of the segment determined by $P(0, 3)$ and $Q(3, -1)$ under the transformation

$M_x \begin{cases} u = 3x + 0y \\ v = 0x + y \end{cases}$. Show the segment and its image on the same diagram.

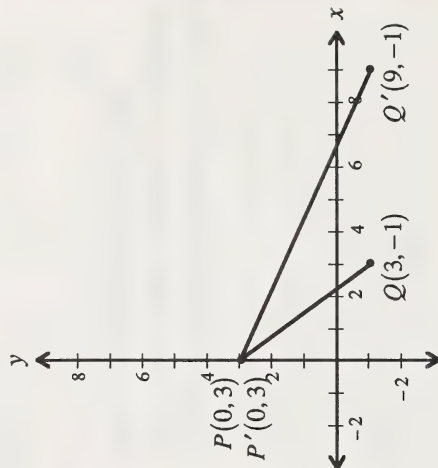
Solution:

$$M_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ y \end{bmatrix}$$

$$M_x \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$M_x \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$$

Therefore, the image is $P'Q'$, where P' is $(0, 3)$ and Q' is $(9, -1)$.



The next expansion is the expansion along the y -axis.

Note that $M_x \begin{cases} u = 3x + 0y \\ v = 0x + y \end{cases}$ can be expressed as $M_x = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

M_x represents the magnification in the x -direction.

The transformation is $M_y \begin{cases} u = x + 0y \\ v = 0x + ky \end{cases}$ or

$$M_y = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}.$$



The image $M_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}$.

This transformation is called magnification in the y -direction.

Example 5

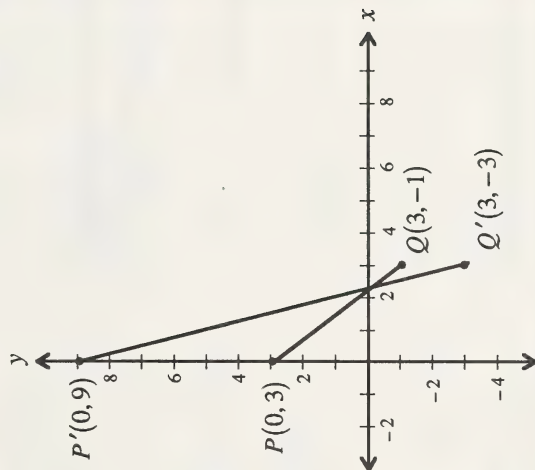
For the transformation $M_y = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, find the image of segment PQ in Example 4. Show this segment and its image on the same diagram.

Solution:

$$M_y = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$M_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x \\ 3y \end{bmatrix}$$



Therefore, P' is $(0, 9)$ and Q' is $(3, -3)$.

$$M_y \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

$$M_y \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

M_y is the magnification in the y -direction.

Another kind of transformation is called **projection** which may be on the x -axis or the y -axis. The projection of a point P onto a point on the x -axis is a perpendicular dropped to the x -axis from that point. Therefore, the projection has the same x -coordinate as the point P , and its y -coordinate is always zero. The transformation

$P_x \begin{cases} u = x + 0y \\ v = 0x + 0y \end{cases}$ or $P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has the effect of projecting a point on the x -axis.

Similar to the projection on the x -axis, the transformation

$P_y \begin{cases} u = 0x + 0y \\ v = 0x + y \end{cases}$ or $P_y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ has the effect of projecting a point on the y -axis.

The image of a point projected on the x -axis is

$$P_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$



The image of a point projected on the y -axis is

$$P_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$

Example 6

If PQ is a segment determined by $P(1, 2)$ and $Q(3, 5)$, find the images of PQ projected on the x -axis and the y -axis. Draw the diagram.

Solution:

Projection on the x -axis:

$$P_x \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P_x \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

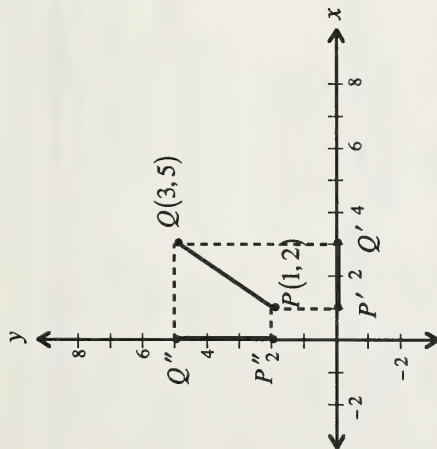
Therefore, the projection of PQ on the x -axis is $P'Q'$ where $P' = (1, 0)$ and $Q' = (3, 0)$.

Projection on the y -axis:

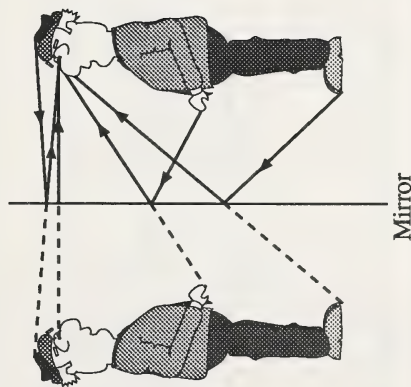
$$P_y \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$P_y \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

Therefore, the projection of PQ on the y -axis is $P''Q''$ where $P'' = (0, 2)$ and $Q'' = (0, 5)$.



When you got up this morning, did you look at your image in a mirror? Did your image appear to be behind the mirror? If you are one metre in front of the mirror, your image seems to be one metre behind it.



You and your image are symmetric with respect to the mirror. If a transformation has the effect of **reflection** in the y-axis, the point and the image under this transformation would be symmetric with respect to the y-axis. The transformation involved is

$$R_y \begin{cases} u = -x + 0y \\ v = 0x + y \end{cases} \text{ or } R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \text{ The image of a}$$

$$\text{point } (x, y) \text{ is } R_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

Similar to reflection in the y-axis, a point and its image under the transformation causing a reflection in the x-axis is symmetric with respect to the x-axis. The transformation causing a reflection in the x-axis is symmetric with respect to the x-axis. The



$$\text{transformation is } R_x \begin{cases} u = x + 0y \\ v = 0x - y \end{cases} \text{ or } R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$\text{The image of a point } (x, y) \text{ is } R_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

Example 7

A segment PQ determined by $P(2, 2)$ and $Q(6, 5)$ is reflected in both the x -axis and the y -axis. Find the two images and draw the diagram.

Solution:

The reflection in the x -axis is as follows:

$$R_x \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \quad \left(\text{since } R_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \right)$$

$$R_x \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

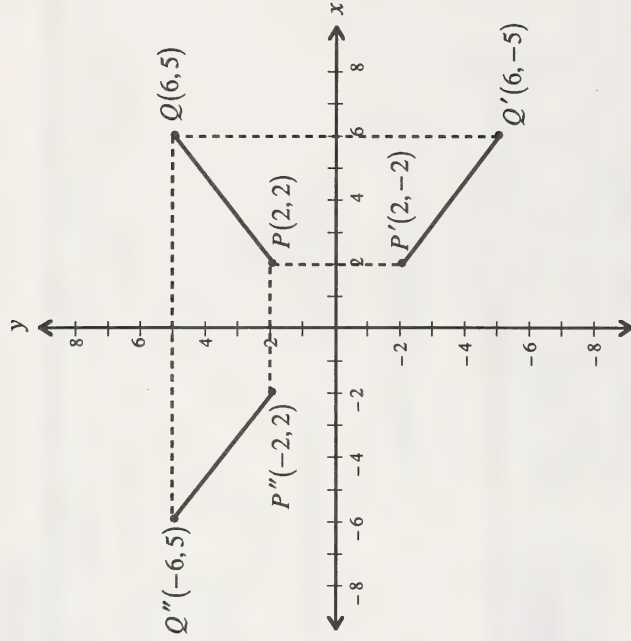
Therefore, image $P'Q'$ is determined by $P'(2, -2)$ and $Q'(6, -5)$.

The reflection in the y -axis is as follows:

$$R_y \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \quad \left(\text{since } R_y \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \right)$$

$$R_y \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

Therefore, image $P''Q''$ is determined by $P''(-2, 2)$ and $Q''(-6, 5)$.



The previous example shows reflections in the x -axis and the y -axis. A point and its image can also be symmetric with respect to the origin. The transformation causing a reflection in the origin is

$$R_o \begin{cases} u = -x + 0y \\ v = 0x - y \end{cases} \text{ or } R_o = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and the image}$$



$$\text{of a point } (x, y) \text{ is } R_o \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}.$$

Therefore, for reflection in the origin, the image of the segment PQ in Example 7 would be

$$R_o \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \text{ and } R_o \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \end{bmatrix}, \text{ and the image of}$$

$$PQ \text{ would be } P'''Q''' \text{ with } P'''(-2, -2) \text{ and } Q'''(-6, -5).$$

The transformations you have studied are special transformations. The last one to be discussed is called **shear**. Shear is the lateral deformation produced in a body. To see shear in action, draw a picture or write a word on the edge of a thick book. (Use an old telephone book.) Push on the top so that the pages slide over. You can see that the picture will be distorted. The distortion will vary depending on how you push the book. The external force that you apply acts like a transformation. A transformation which leaves the y -coordinate of a point unchanged, but shifts each point parallel to the x -axis is called a shear in the x -direction. The amount of shift depends on y , which is the distance from the x -axis, and on a constant of proportionality (a).



An animal



An animal sheared

If a transformation leaves the x -coordinates of a set of points unchanged, but shifts each point parallel to the y -axis, it is called shear in the y -direction. The amount of shift depends on x , which is the distance from the y -axis, and on a constant of proportionality (a).



$$\text{The shear in the } x\text{-direction is } S_x \begin{cases} u = x + ay \\ v = 0x + y \end{cases} \text{ or}$$

$$S_x = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}. \text{ Therefore, the image of a point } (x, y)$$

$$\text{is } S_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}.$$



The shear in the y -direction is

$$S_y \begin{cases} u = x + 0y \\ v = ax + y \end{cases} \text{ or } S_y = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}.$$

Therefore, the image of $P(x, y)$

$$\text{is } \begin{bmatrix} x \\ ax + y \end{bmatrix}.$$

The following example shows shear in action.

Example 8

Under shear in the x -direction $S_x \begin{cases} u = x - 3y \\ v = y \end{cases}$

and shear in the y -direction $S_y \begin{cases} u = x \\ v = 3x + y \end{cases}$, find the images of segment PQ determined by $P(1, 2)$ and $Q(4, 5)$.

Draw the segments and their images on the same graph.

Solution:

Shear in the x -direction:

$$\begin{aligned} \text{Image of } P &= \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -5 \\ 2 \end{bmatrix} = P' \end{aligned}$$

$$\begin{aligned} \text{Image of } Q &= \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 - 3(5) \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -11 \\ 5 \end{bmatrix} = Q' \end{aligned}$$

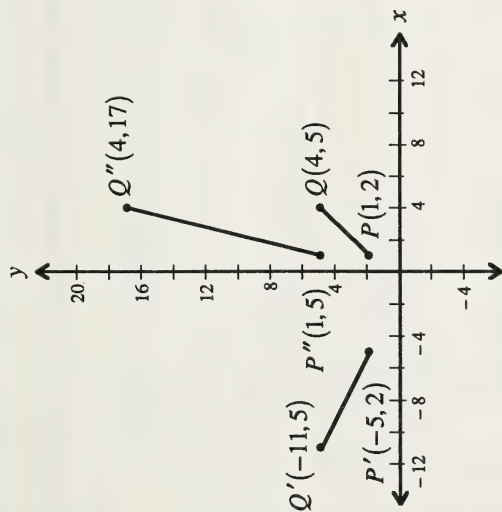
Shear in the y -direction:

$$\begin{aligned} \text{Image of } P &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3(1) + 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 5 \end{bmatrix} = P'' \end{aligned}$$

$$\begin{aligned} \text{Image of } Q &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 3(4) + 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 17 \end{bmatrix} = Q'' \end{aligned}$$

For the shear in the x -direction, -3 is the constant of proportionality.

For the shear in the y -direction, 3 is the constant of proportionality.



So far you have been doing one special transformation each time.

Can two or more special transformations be done in succession?

Yes, the result is called a **compound or product transformation**.



Suppose you transform the point $P(1, 2)$ by the transformation $S_x \begin{cases} u = x + 4y \\ v = y \end{cases}$, causing a shear in the x -direction, followed by $R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, a reflection in the x -axis.

The product of the two transformations is

$$R_x S_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}.$$

The image of $P(1, 2)$ is

$$\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(1) + 4(2) \\ 0(1) + (-1)(2) \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}.$$

Remember that $R_x S_x \neq S_x R_x$.

The following chart is a summary of the special transformations you have learned.

Transformation	Image	Matrix
Identity	$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Zero	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Rotation (R_θ)	$\begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix}$	$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$
Reflection in x-axis (R_x)	$\begin{bmatrix} x \\ -y \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in y-axis (R_y)	$\begin{bmatrix} -x \\ y \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in origin (R_o)	$\begin{bmatrix} -x \\ -y \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
Uniform magnification (M)	$\begin{bmatrix} kx \\ ky \end{bmatrix}$	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

Transformation	Image	Matrix
Magnification in the x -direction (M_x)	$\begin{bmatrix} kx \\ y \end{bmatrix}$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Magnification in the y -direction (M_y)	$\begin{bmatrix} x \\ ky \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Projection on the x -axis (P_x)	$\begin{bmatrix} x \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection on the y -axis (P_y)	$\begin{bmatrix} 0 \\ y \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Shear in the x -direction (S_x)	$\begin{bmatrix} x + ay \\ y \end{bmatrix}$	$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$
Shear in the y -direction (S_y)	$\begin{bmatrix} x \\ ax + y \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

Since you have learned so many special transformations, do the following questions to reinforce what you have learned.

1. Find the matrix that represents a rotation through 120° .
2. Find the matrix corresponding to a rotation of -45° followed by a reflection in the x -axis.
3. Find the image of point $P(5, 4)$ under the transformation which represents a rotation through 30° .
4. Find the image of segment PQ determined by $P(-1, 1)$ and $Q(3, 2)$ under the following transformations.

a. $M_x \begin{cases} u = 3x \\ v = 3y \end{cases}$ b. $M_x \begin{cases} u = 4x + 0y \\ v = 0x + y \end{cases}$

c. $M_y \begin{cases} u = x + 0y \\ v = 0x - 2y \end{cases}$

Draw all segments and their images on the same diagram. Graph paper is provided in **Appendix B**.

5. Find the two images of segment PQ determined by $P(-2, 1)$ and $Q(3, 2)$ if the image of PQ is projected on the x -axis and y -axis. Draw the diagram.

6. A segment PQ determined by $P(1, 1)$ and $Q(5, 5)$ is reflected in the x -axis, the y -axis, and the origin. Find the three images and draw the diagram.

7. A triangle has vertices at $P(0, 0)$, $Q(3, 0)$, and $R(2, 2)$. Find the image of this triangle under the shear $S_x \begin{cases} x' = x + 3y \\ y' = 0x + y \end{cases}$ and draw the diagram.

8. A square has vertices at $A(1, 1)$, $B(1, -1)$, $C(-1, -1)$, and $D(-1, 1)$. Find its image under the shear $S_y \begin{cases} x' = x \\ y' = 3x + y \end{cases}$ and draw the diagram.

9. Find the matrix that is equivalent to a reflection in the origin followed by a rotation through 45° .

10. Find the image of point $P(-3, 4)$ under the uniform magnification followed by a projection on the y -axis. (Note: Let $k = 3$.)



For solutions to **Activity 1**, turn to **Appendix A**, **Topic 4**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.



Extra Help

In this topic you have learned seven special transformations. Each of them can be denoted by a matrix.

Identity Transformation

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This transformation does not really change anything. The image of a point (x, y) is the point itself.

Zero Transformation

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The image of any point (x, y) is always $(0, 0)$.

Rotations

$$R_{\theta} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

This transformation rotates the plane about the origin through an angle θ . For example, if $(3, 3)$ is rotated through an angle of 30° , then the image is as follows:

$$\begin{aligned} R_{30^\circ} \begin{bmatrix} 3 \\ 3 \end{bmatrix} &= \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3\sqrt{3}}{2} - \frac{3}{2} \\ \frac{3}{2} + \frac{3\sqrt{3}}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1.098 \\ 4.098 \end{bmatrix} \end{aligned}$$

If the rotation is a clockwise rotation, then the angle is negative. For example, if $(3, 3)$ is rotated through an angle of -30° , then the image can be calculated as follows:

$$\begin{aligned}
 R_{-30^\circ} \begin{bmatrix} 3 \\ 3 \end{bmatrix} &= \begin{bmatrix} \cos(-30^\circ) & -\sin(-30^\circ) \\ \sin(-30^\circ) & \cos(-30^\circ) \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{3\sqrt{3}}{2} + \frac{1}{2} \\ -\frac{3}{2} + \frac{3\sqrt{3}}{2} \end{bmatrix} \\
 &\doteq \begin{bmatrix} 4.098 \\ 1.098 \end{bmatrix}
 \end{aligned}$$

Magnification

There are three types of magnifications.

- Uniform magnification

$$M = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

This transformation is a uniform expansion or contraction. For example, $M = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ would magnify the length of a segment by three times.

- Magnification in the x-direction

$$M_x = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

The expansion is in the x-direction only. For example,

$$M_x = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \text{ would magnify only the } x\text{-values by three times.}$$

The image of the point $(2, 3)$ will be $(6, 3)$. The y-value remains unchanged.

- Magnification in the y -direction

$$M_y = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

The expansion is along the y -direction only. For example,

$M_y = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ would magnify the y -value by three times. The

image of the point $(2, 3)$ can be obtained by multiplying the y -value by 3 and leaving the x -value unchanged. The image of $(2, 3)$ is $(2, 9)$.

Projections

There are two types of projections.

- Projection on the x -axis

$$P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The image of any point (x, y) is always $(x, 0)$. For example, the image of $(3, 5)$ is $(3, 0)$.

- Projection on the y -axis

$$P_y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The image of any point (x, y) is always $(0, y)$. For example, the image of $(2, 8)$ is $(0, 8)$.

Reflections

There are three types of reflections.

- Reflection in the x -axis

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The image of any point (x, y) is always $(x, -y)$. For example, the image of $(3, 7)$ is $(3, -7)$.

- Reflection in the y -axis

$$R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The image of any point (x, y) is always $(-x, y)$. For example the image of $(5, 3)$ is $(-5, 3)$.

- Reflection in the origin

$$R_o = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The image of any point (x, y) is always $(-x, -y)$. For example, the image of $(3, 7)$ is $(-3, -7)$.

Shear

There are two types of shears.

- Shear in the x -direction

$$S_x = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

The constant of proportionality is a . The image of any point (x, y) is $(x + ay, y)$. The effect of this transformation is to shift every point parallel to the x -axis. (Note that a must be given.)

- Shear in the y -direction

$$S_y = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

The constant of proportionality is a . The image of any point (x, y) is $(x, y + ax)$. The effect of this transformation is to shift every point parallel to the y -axis. (Note that a must be given.)

Now do the following questions.

1. Find the image of the point $(3, 7)$ under the identity transformation.
2. Find the image of the point $(2, 4)$ under the zero transformation.
3. Find the image of the point $(1, 2)$ under the transformation which represents a rotation through 45° .
4. Find the image of the segment determined by $P(1, 2)$ and $Q(3, 4)$ under the following transformations.

a. $M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

b. $M_x = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

c. $M_y = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

5. Find the image of the point $(5, 3)$ projected on the x -axis.
6. Find the image of the point $(5, 3)$ projected on the y -axis.
7. Find the image of the point $(3, 5)$ reflected in the x -axis.

8. Find the image of the point $(3, 5)$ reflected in the y -axis.

9. Find the image of the point $(3, 5)$ reflected in the origin.

10. Find the images of the segment determined by $P(1, 2)$ and $Q(3, 4)$ under the following shears.

a. $S_x = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

b. $S_y = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$



For solutions to **Extra Help**, turn to **Appendix A, Topic 4**.



Extensions

Questions 9 and 10 of the practice exercise in this topic asked you to perform two transformations. Did you do the transformations one by one? In Topic 3 it was mentioned that the product of two transformations was the product of two matrices, and it has the effect of two mappings. Now you may want to carry out these successive transformations by matrix multiplication.

Example 9

Find the image of the point $(5, 4)$ rotated counterclockwise through an angle of 45° and reflected in the x -axis.

Solution:

The transformations are $R_{\frac{\pi}{4}} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}$ and

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The image can be obtained by multiplying the matrices.

$$\begin{aligned}
 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{2}} - \frac{4}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} + \frac{4}{\sqrt{2}} \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{9}{\sqrt{2}} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{9}{\sqrt{2}} \end{bmatrix}
 \end{aligned}$$

Now do the following questions.

1. Find the image of the point $(3, 1)$ rotated counterclockwise through an angle of 30° and projected on the y -axis.
2. Show that the image of the point $(2, 5)$ reflected in the x -axis and then the y -axis is equivalent to the image of the point reflected in the origin.



For solutions to Extensions, turn to **Appendix A, Topic 4**.

Unit Summary



What You Have Learned

In your study of matrices and linear transformations you have learned the following:

- An $m \times n$ matrix is a matrix of m rows and n columns. This matrix has dimensions $m \times n$.
- It is possible to add two matrices if they have the same dimensions by adding their corresponding elements.
- To multiply a matrix by a constant, you multiply every element by the same constant.
- The closure property, associative property, and commutative property hold for all matrices under the operation of addition.
- The additive identity in the set of matrices is a zero matrix.
- Multiplication of matrices is a row-by-column multiplication, and it can only occur when the number of columns of the first matrix equals the number of rows of the second matrix.
- Matrix multiplication is associative and distributive, but not commutative.

Unit Summary

- The multiplication identity in the set of matrices is a matrix whose principal diagonal from upper left to lower right has elements of 1, while all other elements are zeros.
- A linear transformation describes a particular type of correspondence between two sets.
- An inverse transformation will undo what a transformation has done.
- The image of a point is the product of two matrices.
- The following are special transformations:
 - Identity transformation

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Zero transformation

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Unit Summary

– Rotations

$$R_{\theta} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

– Magnifications

$$M = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, M_x = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, M_y = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

– Projection

$$P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, P_y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

– Reflections

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, R_o = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

– Shears

$$S_x = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, S_y = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

- Two transformations in succession are the product of these two transformations.

You are now ready to
complete the **Unit Assignment**.

Appendices



Appendix A Solutions

Review

Topic 1 Operations Defined on Matrices

Topic 2 Linear Transformation and Its
Inverse

Topic 3 Product of Two Transformations

Topic 4 Special Transformations



Appendix B Graphing Material

Graph Paper



Appendix A Solutions



Review

1. $3x + 5y - 6z = 1$ (1)

$x + y - z = 2$ (2)

$x - 2y + z = 3$ (3)

(2) - (3): $3y - 2z = -1$ (4)

(1) - 3 × (2): $2y - 3z = -5$ (5)

$3 \times (4) - 2 \times (5):$ $5y = 7$

$y = \frac{7}{5}$

Substitute $y = \frac{7}{5}$ in (4).

$3\left(\frac{7}{5}\right) - 2z = -1$

$-2z = -1 - \frac{21}{5}$

$z = \frac{26}{10}$

$= \frac{13}{5}$

Substitute $y = \frac{7}{5}$ and $z = \frac{13}{5}$ in (3).

$x - 2\left(\frac{7}{5}\right) + \left(\frac{13}{5}\right) = 3$

$x - \frac{1}{5} = 3$

$x = \frac{16}{5}$

2. Augmented matrix = $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 4 \\ 2 & 1 & -1 & 7 \\ 1 & 2 & 1 & 5 \end{array} \right]$

(2) - 2 × (1): $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 4 \\ 0 & 3 & -5 & -1 \\ 1 & 2 & 1 & -1 \end{array} \right]$

(1) - (3): $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 4 \\ 0 & 3 & -5 & -1 \\ 0 & -3 & 1 & -1 \end{array} \right]$

(2) + (3): $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 4 \\ 0 & 3 & -5 & -1 \\ 0 & 0 & -4 & -2 \end{array} \right]$

$\frac{1}{3} \times (2):$ $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 4 \\ 0 & 1 & -\frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} \end{array} \right]$

$-\frac{1}{4} \times (3):$

$$\begin{aligned}
 3. \quad x - y + 2z &= 4 & \textcircled{1} \\
 y - \frac{5}{3}z &= -\frac{1}{3} & \textcircled{2} \\
 z &= \frac{1}{2} & \textcircled{3}
 \end{aligned}$$

Substitute $\textcircled{3}$ in $\textcircled{2}$.

$$\begin{aligned}
 y &= -\frac{1}{3} + \frac{5}{3}\left(\frac{1}{2}\right) \\
 &= -\frac{1}{3} + \frac{5}{6} \\
 &= \frac{3}{6} \\
 &= \frac{1}{2}
 \end{aligned}$$

Substitute $z = \frac{1}{2}$ and $y = \frac{1}{2}$ in $\textcircled{1}$.

$$\begin{aligned}
 x - \frac{1}{2} + 2\left(\frac{1}{2}\right) &= 4 \\
 x + \frac{1}{2} &= 4 \\
 x &= \frac{9}{2}
 \end{aligned}$$

Therefore, the solution is $x = \frac{9}{2}$, $y = \frac{1}{2}$, and $z = \frac{1}{2}$.



Exploring Topic 1

Activity 1

Add and subtract matrices.

$$\begin{aligned}
 1. \quad \text{a. } a_{21} &= 5 & \text{b. } a_{21} &= 1 \\
 a_{12} &= 7 & a_{12} &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{c. } a_{21} &= 2 \\
 a_{12} &= 1 \\
 a_{32} &= -3
 \end{aligned}$$

$$2. \quad \text{a. } 2 \times 2 \qquad \text{b. } 2 \times 3$$

$$\text{c. } 3 \times 2$$

$$\begin{aligned}
 3. \quad a &= 5 \\
 b &= -1
 \end{aligned}$$

$$\begin{aligned} 4. \quad a. \quad 4+2(-3) &= 4-6 \\ &= -2 \end{aligned}$$

$$\begin{aligned} b. \quad (-5)(-2) + (-6)(-1) &= 10+6 \\ &= 16 \end{aligned}$$

$$\begin{aligned} 5. \quad A+B &= \begin{bmatrix} 2+0 & 3+0 & -5+4 \\ 1+(-1) & -6+(-3) & 4+(-2) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 & -1 \\ 0 & -9 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 6. \quad a. \quad A+B &= \begin{bmatrix} 0+2 & -3+(-1) \\ 1+1 & 4+3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -4 \\ 2 & 7 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} b. \quad B-C &= \begin{bmatrix} 2-2 & -1-0 \\ 1-5 & 3-(-3) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ -4 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} c. \quad -3B &= -3 \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 3 \\ -3 & -9 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} d. \quad \frac{1}{2}A &= \frac{1}{2} \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\frac{3}{2} \\ \frac{1}{2} & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} e. \quad A-[3B+C] &= \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix} - \left(3 \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 5 & -3 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 6+2 & -3+0 \\ 3+5 & 9-3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 8 & -3 \\ 8 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 0-8 & -3-(-3) \\ 1-8 & 4-6 \end{bmatrix} \\ &= \begin{bmatrix} -8 & 0 \\ -7 & -2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 7. \text{ a. } A+B &= \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} 5+0 & 3+2 \\ -2+0 & 1+(-1) \end{bmatrix} \\
 &= \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 B+A &= \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0+5 & 2+3 \\ 0+(-2) & (-1)+1 \end{bmatrix} \\
 &= \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}
 \end{aligned}$$

Therefore, $A+B=B+A$.

$$\begin{aligned}
 \text{b. } (A+B)+C &= \left(\begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \right) + \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 6 & 3 \\ -2 & 3 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 A+(B+C) &= \left(\begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \right) + \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 5 & 3 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 6 & 3 \\ -2 & 3 \end{bmatrix}
 \end{aligned}$$

Therefore, $(A+B)+C=A+(B+C)$.

$$\begin{aligned}
 8. \quad 2A-B-3C &= 2 \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix} - 3 \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & -2 \end{bmatrix} \\
 &= \begin{bmatrix} 6 & 4 & 0 \\ 2 & 10 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 3 & -9 \\ 0 & 3 & -6 \end{bmatrix} \\
 &= \begin{bmatrix} 6-1-6 & 4-3-3 & 0-2-(-9) \\ 2-1-0 & 10-(-1)-3 & -4-0-(-6) \end{bmatrix} \\
 &= \begin{bmatrix} -1 & -2 & 7 \\ 1 & 8 & 2 \end{bmatrix}
 \end{aligned}$$

Activity 2

Multiply matrices.

$$\begin{aligned}
 1. \quad AB &= \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 3(1)+0(3) & 3(4)+0(2) \\ -1(1)+2(3) & -1(4)+2(2) \end{bmatrix} \\
 &= \begin{bmatrix} 3 & 12 \\ 5 & 0 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 1(3)+4(-1) & 1(0)+4(2) \\ 3(3)+2(-1) & 3(0)+2(2) \end{bmatrix} \\
 &= \begin{bmatrix} -1 & 8 \\ 7 & 4 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad AB &= \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 0(3)+1(1)+2(-1) & 0(2)+1(4)+2(5) \end{bmatrix} \\
 &= \begin{bmatrix} -1 & 14 \end{bmatrix}
 \end{aligned}$$

$$BA = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad (\text{impossible})$$

$$\begin{aligned}
 3. \quad AB &= \begin{bmatrix} 2 & 1 & -3 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & -1 & 5 \\ -2 & 1 & -4 & 7 \end{bmatrix} \\
 &= \begin{bmatrix} 2(1)+1(0)+(-3)(-2) & 2(2)+1(3)+(-3)(1) & 2(3)+1(-1)+(-3)(-4) & 2(4)+1(5)+(-3)(7) \\ 0(1)+(-1)(0)+(-2)(-2) & 0(2)+(-1)(3)+(-2)(1) & 0(3)+(-1)(-1)+(-2)(-4) & 0(4)+(-1)(5)+(-2)(7) \end{bmatrix} \\
 &= \begin{bmatrix} 8 & 4 & 17 & -8 \\ 4 & -5 & 9 & -19 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad A(BC) &= \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 0 & 1 \end{bmatrix} \right) \\
 &= \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -10 & 9 \end{bmatrix} \\
 &= \begin{bmatrix} -25 & 20 \\ 15 & -6 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 (AB)C &= \left(\begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \right) \begin{bmatrix} 5 & -3 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} -5 & 5 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} -25 & 20 \\ 15 & -6 \end{bmatrix}
 \end{aligned}$$

Therefore, $A(BC) = (AB)C$.

$$\begin{aligned}
 5. \quad A(B+C) &= \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ -2 & 4 \end{bmatrix} \right) \\
 &= \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ -1 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 2 & 8 \\ -2 & 8 \end{bmatrix}
 \end{aligned}$$

$$6. \quad A^2 = AA$$

$$\begin{aligned}
 &= \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 9 \\ 0 & 4 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad AB &= \begin{bmatrix} 2(5)+3(1)+(-1)(3) & 2(2)+3(0)+(-1)(4) & 2(1)+3(6)+(-1)(1) \\ 1(5)+5(1)+(2)(3) & 1(2)+5(0)+2(4) & 1(1)+5(6)+(2)(1) \end{bmatrix} \\
 &= \begin{bmatrix} 10 & 0 & 19 \\ 16 & 10 & 33 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 AB+AC &= \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ -2 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 5 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 9 \\ -4 & 8 \end{bmatrix} \\
 &= \begin{bmatrix} 2 & 8 \\ -2 & 8 \end{bmatrix}
 \end{aligned}$$

Therefore, $A(B+C) = AB+AC$.

$$8. A^2 = AA$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 7 \\ 14 & 18 \end{bmatrix}$$

$$B^2 = BB$$

$$= \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 12 & 8 \end{bmatrix}$$

$$A^2 + B^2 = \begin{bmatrix} 11 & 7 \\ 14 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 12 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 10 \\ 26 & 26 \end{bmatrix}$$

Extra Help

$$1. A+B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 5 \\ 3 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 3+1 & 0+5 \\ 2+3 & 0+(-2) & 4+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 5 \\ 5 & -2 & 6 \end{bmatrix}$$

$$2. A-B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 8 \\ 4 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 3-0 & 1-8 \\ 5-4 & 2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -7 \\ 1 & 11 \end{bmatrix}$$

$$3. \quad A \times B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 & -1 \\ 3 & 8 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3(0)+2(3) & 3(2)+2(8) & 3(1)+2(1) & 3(-1)+2(-2) \\ 1(0)+0(3) & 1(2)+0(8) & 1(1)+0(1) & 1(-1)+0(-2) \\ 5(0)+(-1)(3) & 5(2)+(-1)(8) & 5(1)+(-1)(1) & 5(-1)+(-1)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 22 & 5 & -7 \\ 0 & 2 & 1 & -1 \\ -3 & 2 & 4 & -3 \end{bmatrix}$$

Extensions

There are many solutions. One example is $A = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 9 & 7 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 0 \\ 8 & 4 \end{bmatrix}$.

$$AB = \begin{bmatrix} 12 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 12 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore, $AB = AC$, but $B \neq C$.



Exploring Topic 2

Activity 1

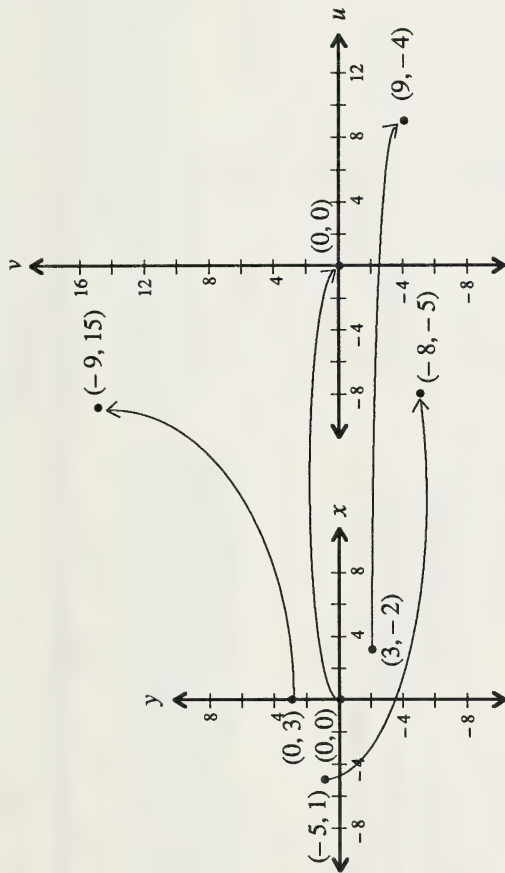
Define linear transformation, onto transformation, and into transformation.

1. a. $T \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$ $\begin{cases} u = (3) - 3(-2) = 9 \\ v = 2(3) + 5(-2) = -4 \end{cases}$

b. $T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{cases} u = 0 \\ v = 0 \end{cases}$

c. $T \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 15 \end{bmatrix}$ $\begin{cases} u = 0 - 3(3) = -9 \\ v = 2(0) + 5(3) = 15 \end{cases}$

d. $T \begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$ $\begin{cases} u = -5 - 3(1) = -8 \\ v = 2(-5) + 5(1) = -5 \end{cases}$



$$2. \quad v \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2(-2) \\ -3(-3) \end{bmatrix} \\ = \begin{bmatrix} -4 \\ 9 \end{bmatrix}$$

3. a. Since the images are always positive real numbers, then this is only part of set B . Therefore, this is an into mapping.
- b. This is an onto mapping.
4. $-1 = 3x - 2y$ (1) (Note that $u = -1$ and $v = 3$.)
 $3 = x + y$ (2)

$$\textcircled{4} + 2 \times \textcircled{1}: 5 = 5x$$

$$x = 1$$

$$y = 3 - x$$

$$= 3 - 1$$

$$= 2$$

Therefore, $(x, y) = (1, 2)$.

5. S is a transformation of the rs -plane. It produces an image for every point (r, s) . Every point in the mn -plane is the image of a unique point in the rs -plane. It is an onto mapping.

$$6. \begin{cases} u = 3(2) - 5(1) = 1 \\ v = 2 + 3(1) = 5 \end{cases}$$

$$\text{Therefore, } S \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

7. All images lie on the line $v = \frac{1}{2}u$ in the uv -plane. Other points on the uv -plane are not images of (x, y) . Therefore, this is a mapping of the xy -plane into the uv -plane.

Activity 2

Determine linear transformations using matrix multiplication.

$$\begin{aligned} 1. \quad W \begin{bmatrix} 0 \\ 2 \end{bmatrix} &= \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 - 3(2) \\ 2(0) - 6(2) \end{bmatrix} \\ &= \begin{bmatrix} -6 \\ -12 \end{bmatrix} \end{aligned}$$

Therefore, $u = -6$ and $v = -12$.

Thus, $x - 3y = -6$. Let $y = k$; then $x = -6 + 3k$.

The set of points whose image under W is $W \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is

$$\{(-6 + 3k, k) \mid k \in R\}.$$

$$2. \begin{cases} -2r - s = -1 & \textcircled{1} \\ 3s = 3 & \textcircled{2} \end{cases}$$

From $\textcircled{2}$, $s = 1$.

Substitute $s = 1$ into $\textcircled{1}$.

$$-2r - 1 = -1$$

$$r = 0$$

Therefore, $(r, s) = (0, 1)$.

3. a. $T = \begin{bmatrix} -1 & -5 \\ 3 & 7 \end{bmatrix}$

b. $S = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$

4. a. $T \begin{cases} u = 3x - 2y \\ v = 5y \end{cases}$

5. a. $T \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$
 $= \begin{bmatrix} 1(3) + (-3)(-2) \\ 2(3) + 5(-2) \end{bmatrix}$
 $= \begin{bmatrix} 9 \\ -4 \end{bmatrix}$

b. $T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
 $= \begin{bmatrix} 1(0) + (-3)(0) \\ 2(0) + 5(0) \end{bmatrix}$
 $= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

c. $T \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$
 $= \begin{bmatrix} 1(0) + (-3)(3) \\ 2(0) + 5(3) \end{bmatrix}$
 $= \begin{bmatrix} -9 \\ 15 \end{bmatrix}$

d. $T \begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \end{bmatrix}$
 $= \begin{bmatrix} 1(-5) + (-3)(1) \\ 2(-5) + 5(1) \end{bmatrix}$
 $= \begin{bmatrix} -8 \\ -5 \end{bmatrix}$

Activity 3

Determine inverse transformations.

1. a. $T^{-1} \begin{cases} x = \frac{d}{ad-bc}u - \frac{b}{ad-bc}v \\ y = \frac{-c}{ad-bc}u + \frac{a}{ad-bc}v \end{cases}$

$ad-bc = (-5)(-1) - 1(1)$
 $= 4$

$\therefore T^{-1} \begin{cases} x = \frac{-1}{4}u - \frac{1}{4}v \\ y = \frac{-1}{4}u - \frac{5}{4}v \end{cases}$

b. $a = -2, b = 0, c = 0, d = 5$

$ad-bc = (-2)(5) - 0 = -10$

$\therefore A^{-1} \begin{cases} x = \frac{5}{-10}u - 0 = -\frac{1}{2}u \\ y = \frac{0}{-10}u + \frac{-2}{-10}v = \frac{1}{5}v \end{cases}$

c. $a = 3, b = -2, c = 0, d = 0$

$ad-bc = 0 - 0 = 0$

Therefore, there is no inverse transformation.

d. $a = -1, b = 2, c = 2, d = -4$

$$ad - bc = (-1)(-4) - (2)(2) = 0$$

Therefore, there is no inverse transformation.

2. $a = 1, b = 3, c = -2, d = 1$

$$ad - bc = 1(1) - 3(-2) = 7$$

$$r = \frac{(1)(-1) - (3)(2)}{7} \qquad s = \frac{-(-2)(-1) + (1)(2)}{7}$$

$$= \frac{-7}{7} = \frac{0}{7} = 0$$

$$= -1 = 0$$

Therefore, $(r, s) = (-1, 0)$.

3. a. $a = -1, b = -3, c = 1, d = -1$

$$ad - bc = (-1)(-1) - (-3)(1) = 4$$

$$x = \frac{du - bv}{ad - bc}$$

$$= \frac{(-1)(-2) - (-3)(-3)}{4}$$

$$= \frac{-7}{4}$$

$$y = \frac{-cu + av}{ad - bc}$$

$$= \frac{-(-1)(-2) + (-1)(-3)}{4}$$

$$= \frac{5}{4}$$

Therefore, $(x, y) = \left(-\frac{7}{4}, \frac{5}{4}\right)$.

b. $x = \frac{(-1)(0) - (-3)(0)}{4} = 0$

$$y = \frac{-(1)0 + (-1)(0)}{4} = 0$$

Therefore, $(x, y) = (0, 0)$.

Extra Help

1. a. $u = (-2) - 2(-3) = 4$

$$v = 5(-2) + (-3) = -13$$

$$\begin{aligned} \text{b. } \begin{bmatrix} 1 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} &= \begin{bmatrix} 1(-2) + (-2)(-3) \\ 5(-2) + 1(-3) \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -13 \end{bmatrix} \end{aligned}$$

2. $a = 1, b = -2, c = 5, d = 1$

$$x = \frac{d}{ad - bc}u - \frac{b}{ad - bc}v$$

$$= \frac{1}{(1)(1) - (-2)(5)}(-1) - \frac{(-2)}{(1)(1) - (-2)(5)}(0)$$

$$= -\frac{1}{11}$$

$$y = \frac{-c}{ad-bc}u + \frac{a}{ad-bc}v$$

$$= \frac{-5}{11}(-1) + \frac{1}{11}(0)$$

$$= \frac{5}{11}$$

Therefore, $(x, y) = (-\frac{1}{11}, \frac{5}{11})$.

3. $a=0, b=2, c=1, d=-1$
 $ad-bc = 0(-1) - 2(1) = -2$
- $$T^{-1} \begin{cases} x = \frac{-1}{-2}u - \frac{2}{-2}v = \frac{1}{2}u + v \\ y = \frac{-1}{-2}u + \frac{0}{-2}v = \frac{1}{2}u \end{cases}$$

Extensions

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$$

Show how the matrix for A^{-1} is determined.

$$A \begin{cases} u = y \\ v = 2x + 3y \end{cases}$$

$$a=0, b=1, c=2, d=3$$

$$A^{-1} \begin{cases} x = \frac{du}{ad-bc} - \frac{bv}{ad-bc} \\ y = \frac{-cu}{ad-bc} + \frac{av}{ad-bc} \end{cases}$$

$$x = \frac{3u}{-2} - \frac{1v}{-2}$$

$$= \frac{-3u}{2} + \frac{1}{2}v$$

$$y = \frac{-2u}{-2} + \frac{0v}{-2} = u$$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 7 & 4 \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{7}{12} & \frac{1}{4} \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ -\frac{7}{12} & \frac{1}{4} \end{bmatrix}$$

Therefore, $(AB)^{-1} = B^{-1}A^{-1}$.



Exploring Topic 3

Activity 1

Define the product of two transformations.

$$1. \quad T \begin{cases} u = 2x - 3y \\ v = x - 3y \end{cases}, \quad S \begin{cases} m = 3u - v \\ n = u + v \end{cases}$$

$$\begin{aligned} \text{a. } T \begin{bmatrix} -2 \\ 3 \end{bmatrix} &= \begin{bmatrix} 2 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -4 - 9 \\ -2 - 9 \end{bmatrix} \\ &= \begin{bmatrix} -13 \\ -11 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} ST \begin{bmatrix} -2 \\ 3 \end{bmatrix} &= S \begin{bmatrix} -13 \\ -11 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -13 \\ -11 \end{bmatrix} \\ &= \begin{bmatrix} -39 + 11 \\ -13 + (-11) \end{bmatrix} \\ &= \begin{bmatrix} -28 \\ -24 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{b. } ST \begin{bmatrix} -1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2(-1) + 0 \\ 1(-1) + 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 + 1 \\ -2 - 1 \end{bmatrix} \\ &= \begin{bmatrix} -5 \\ -3 \end{bmatrix} \end{aligned}$$

$$2. \begin{cases} u = x - 4y \\ v = 2x - y \end{cases}, \begin{cases} m = u - 2v \\ n = -u + v \end{cases}$$

$$A \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1(2) + (-4)(-3) \\ 2(2) + (-1)(-3) \end{bmatrix} = \begin{bmatrix} 14 \\ 7 \end{bmatrix}$$

$$B \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1(2) + (-2)(-3) \\ (-1)(2) + 1(-3) \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

$$B \left(A \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right) = B \begin{bmatrix} 14 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 14 \\ 7 \end{bmatrix} = \begin{bmatrix} 1(14) + (-2)(7) \\ (-1)(14) + 1(7) \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \end{bmatrix}$$

Extra Help

$$1. A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$BA \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} -18 \\ 14 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$$

$$BA \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ -9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -18 \end{bmatrix}$$

Extensions

$$A = \begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$$

$$ABC \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 18 \end{bmatrix}$$

$$= \begin{bmatrix} -18 \\ -18 \end{bmatrix}$$



Exploring Topic 4

Activity 1

Explain transformations dealing with the identity transformation, the zero transformation, rotations, magnifications, projections, reflections, and shears.

$$1. R_{\theta} = \begin{bmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$2. R_{-\frac{\pi}{4}} = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$R_x R_{-\frac{\pi}{4}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$3. R_{\frac{\pi}{6}} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$R_{\frac{\pi}{6}} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{5\sqrt{3}}{2} - 2 \\ \frac{5}{2} + 2\sqrt{3} \end{bmatrix}$$

$$4. \text{ a. Image of } P = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

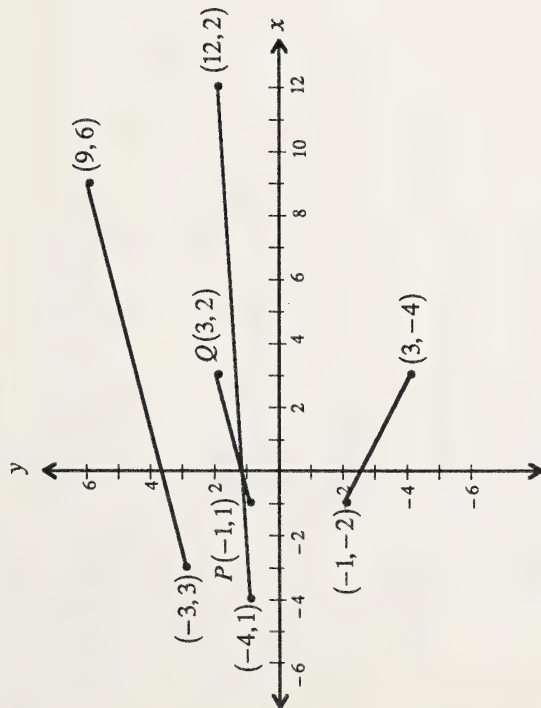
$$\text{Image of } Q = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 27 \\ 18 \end{bmatrix}$$

$$\text{ b. Image of } P = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$\text{Image of } Q = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ 2 \end{bmatrix} = \begin{bmatrix} 48 \\ 2 \end{bmatrix}$$

c. Image of $P = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

Image of $Q = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$



5. Projection on the x-axis:

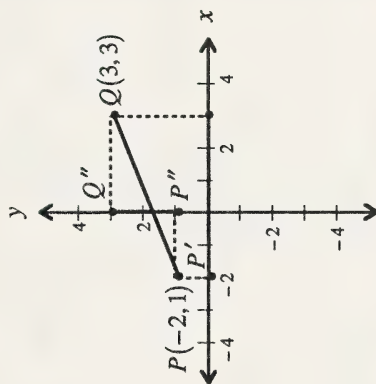
Image of $P = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} = P'$

Image of $Q = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = Q'$

Projection on the y-axis:

Image of $P = \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = P''$

Image of $Q = \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = Q''$



6. Reflection in the x-axis:

$$\text{Image of } P = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = P'$$

$$\text{Image of } Q = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} = Q'$$

Reflection in the y-axis:

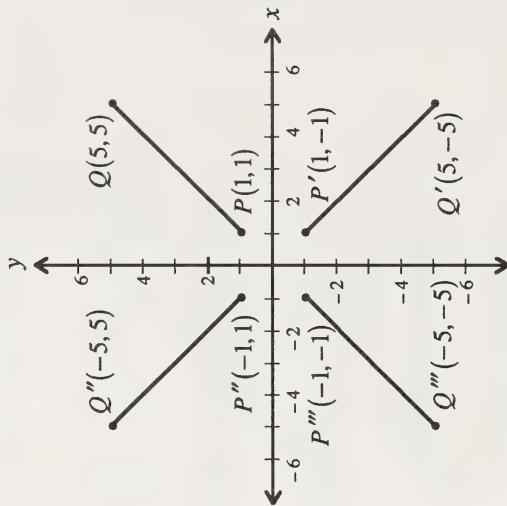
$$\text{Image of } P = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = P''$$

$$\text{Image of } Q = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} = Q''$$

Reflection in the origin:

$$\text{Image of } P = \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = P'''$$

$$\text{Image of } Q = \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix} = Q'''$$

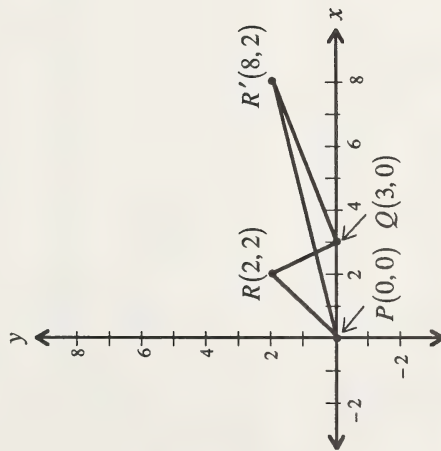


7. $P(0,0)$, $Q(3,0)$, $R(2,2)$

$$\text{Image of } P = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Image of } Q = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\text{Image of } R = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+3(2) \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix} = R'$$



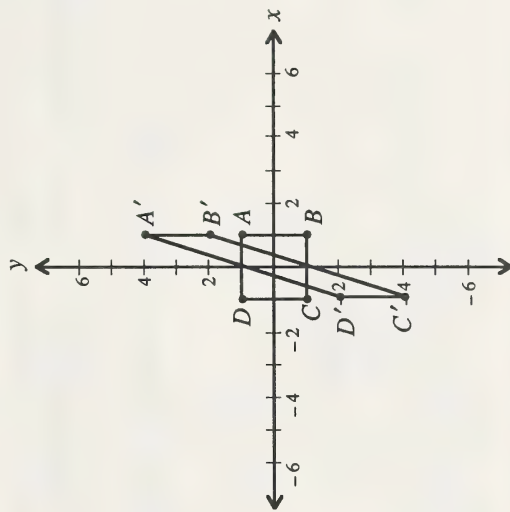
8. $A(1,1)$, $B(1,-1)$, $C(-1,-1)$, $D(-1,1)$

$$\text{Image of } A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = A'$$

$$\text{Image of } B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = B'$$

$$\text{Image of } C = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix} = C'$$

$$\text{Image of } D = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = D'$$



$$\begin{aligned}
 9. R_{\frac{\pi}{4}} R_0 &= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} -x \\ -y \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -x \\ -y \end{bmatrix} \\
 &= \begin{bmatrix} \frac{-x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \\ \frac{-x}{\sqrt{2}} - \frac{y}{\sqrt{2}} \end{bmatrix}
 \end{aligned}$$

10. $P(-3, 4), k=3$

$$\begin{aligned}
 P_y M \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3x \\ 3y \end{bmatrix} \\
 &= \begin{bmatrix} 0 \\ 3y \end{bmatrix}
 \end{aligned}$$

Therefore, $P_y M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$.

Extra Help

1. The image of the point is (3, 7).

2. The image of the point is (0, 0).

$$\begin{aligned}
 3. R_{\frac{\pi}{4}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix}
 \end{aligned}$$

Therefore, the image of (1, 2) is $(-\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}})$.

4. a. The image of PQ is $P'Q'$, where P' is $(1 \times 4, 2 \times 4) = (4, 8)$ and Q' is $(3 \times 4, 4 \times 4) = (12, 16)$.
 b. The image of PQ is $P'Q'$, where P' is $(3 \times 1, 2) = (3, 2)$ and Q' is $(3 \times 3, 4) = (9, 4)$.
 c. The image of PQ is $P'Q'$, where P' is $(1, 2 \times 3) = (1, 6)$ and Q' is $(3, 4 \times 3) = (3, 12)$.
5. The image of the point (5, 3) projected on the x-axis is (5, 0).
6. The image of the point (5, 3) projected on the y-axis is (0, 3).

7. The image of the point $(3, 5)$ reflected in the x -axis is $(3, -5)$.

8. The image of the point $(3, 5)$ reflected in the y -axis is $(-3, 5)$.

9. The image of the point $(3, 5)$ reflected in the origin is $(-3, -5)$.

10. a. The image of $(1, 2)$ is $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$.

The image of $(3, 4)$ is $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 23 \\ 4 \end{bmatrix}$.

Therefore, the image of PQ is $P'Q'$, where P' is $(11, 2)$ and Q' is $(23, 4)$.

b. The image of $(1, 2)$ is $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

The image of $(3, 4)$ is $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \end{bmatrix}$.

Therefore, the image of PQ is $P'Q'$, where P' is $(1, 6)$ and Q' is $(3, 16)$.

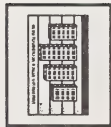
Extensions

$$\begin{aligned}
 1. \quad & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3\sqrt{3}}{2} - \frac{1}{2} \\ \frac{3}{2} + \frac{\sqrt{3}}{2} \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3\sqrt{3}-1}{2} \\ \frac{3+\sqrt{3}}{2} \end{bmatrix} \\
 &= \begin{bmatrix} 0 \\ \frac{3+\sqrt{3}}{2} \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \\
 &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} \\
 &= \begin{bmatrix} -2 \\ -5 \end{bmatrix}
 \end{aligned}$$

Since $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$, the image of the point $(2, 5)$ is

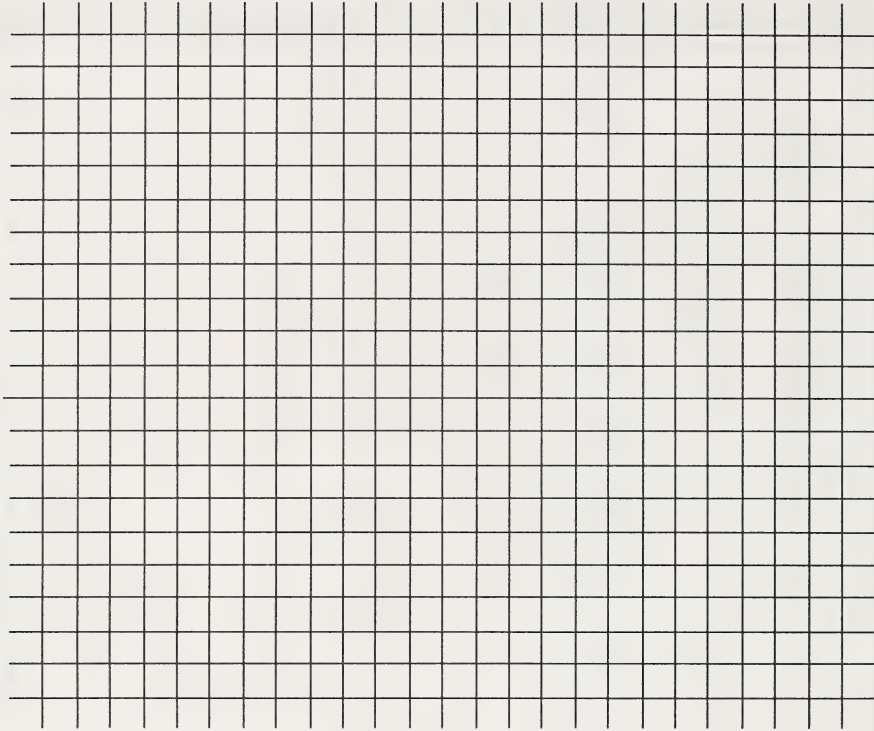
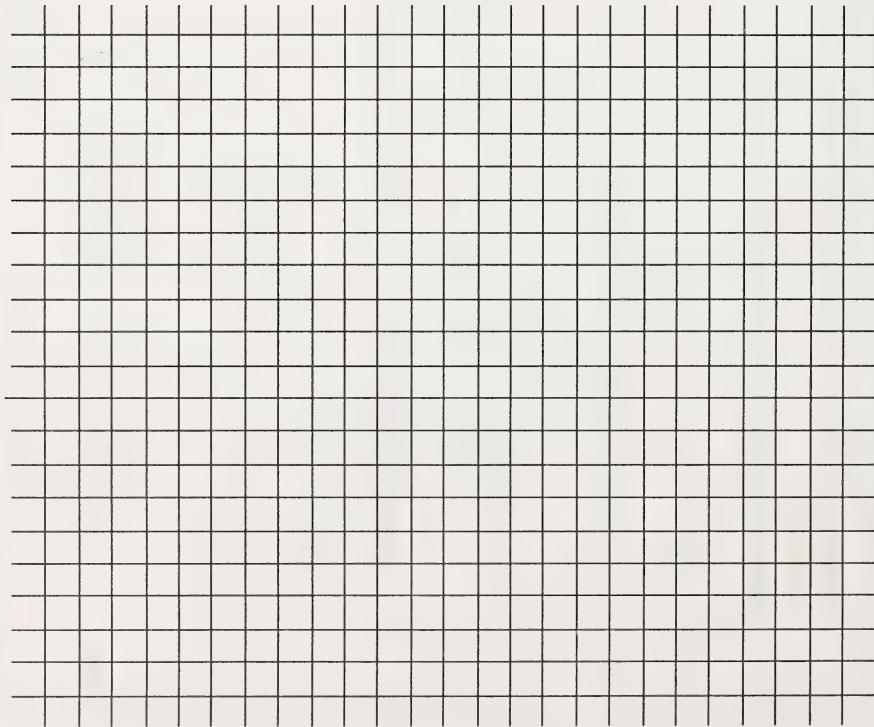
reflected in the x -axis; then, the y -axis is equivalent to the image of the point reflected in the origin.

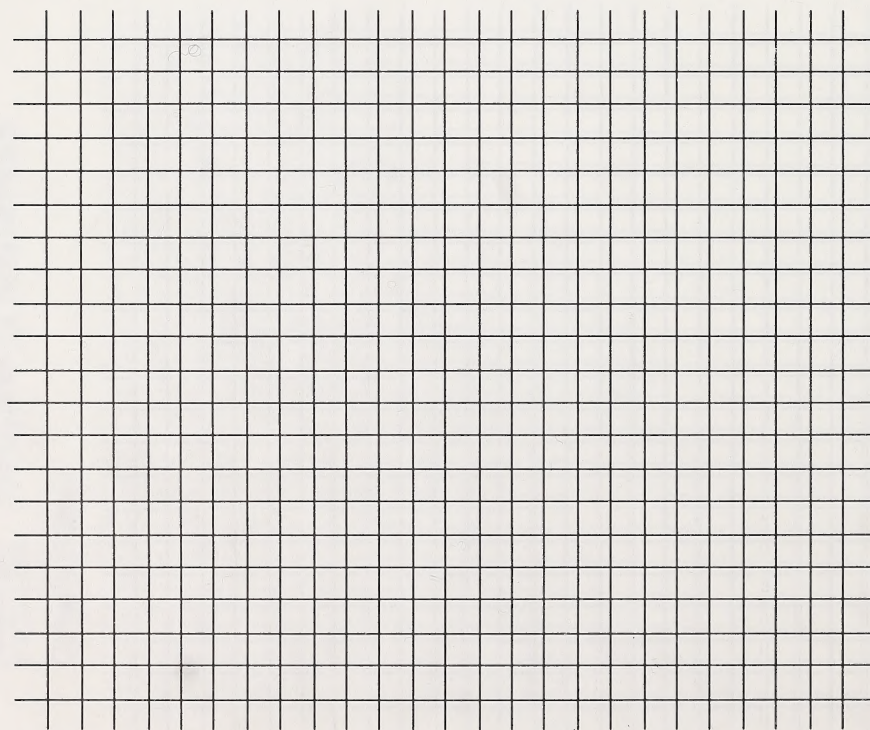
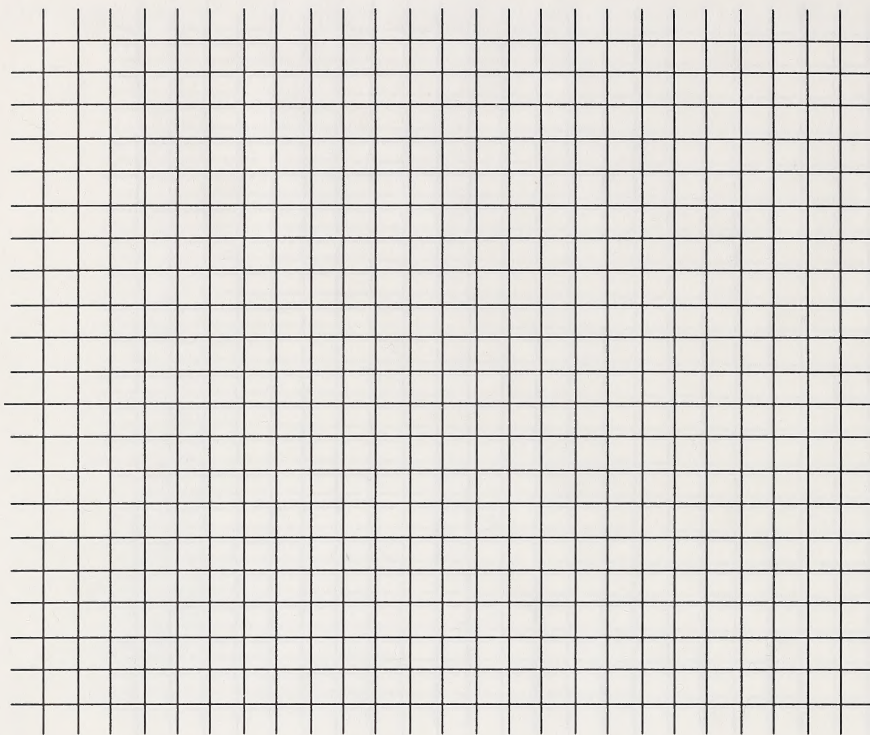


Appendix B

Graphing Materials

Graph Paper





N.L.C. - B.N.C.



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Mathematics 31

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L.R.D.C.
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SECOND EDITION
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